Preservation Value in Socio-Ecological Systems

Abstract

We develop a model that reveals the preservation value of maintaining connectivity within a socio-ecological system (SES). By means of a multiplex network, built from the layers composing the sustainability Venn diagram, we define two measures of preservation value of inter- and intra-layer connections. Most policymakers and researchers have tended to assume that all elements within an SES are unconditionally connected, which implied that dimensions of sustainability functioned unhindered. We must instead explicitly explore how connectivity is operational. Given explicit threats to connectivity, we show under which conditions connectivity is valuable and should be preserved. This implies that policies aimed at sustainability should focus on explicitly addressing connections and disconnections. Using numerical simulations, our results suggest that the preservation value of the SES topological structure is greatest when we secure the connectivity of inter-layer connections.

Keywords: SES, sustainability, connectivity value, spacetime discounting

1 Introduction

Since the famous Bruntland report, people have promoted the idea of sustainability as a noble goal that reflects the need to account for how humans affect nature, and how nature affects humans. The report notably put an emphasis on policies aimed at reducing economic disparities, social exclusion, and environmental degradation (Tomlinson, 1987; Elkington, 1998). The three overlapping realms of economy, society, and environment – structured in the form of a Venn diagram (Mebratu, 1998; Lozano, 2008; Bell, 2011) – then became the most common representation of sustainability, for the reason that policy approaches could be mapped onto the sustainability Venn diagram (Levett, 1998).

Capturing sustainability is challenge, however, due to the many connections and feedbacks between and within social and ecology systems. One tool to capture these connections are polycentric socio-ecological system (SES) models. Those have proved to be a useful organizing device to capture these links and feedbacks between humans and their environment (Ostrom, 2009). By taking into account diverse information flow capabilities (Ostrom, 2010), observers argued that these models describe better the dynamics of interactions between man and nature (Waltner-Toews et al., 2008). A typical SES is composed of anthropogenic and natural elements interacting through temporal, spatial, and organizational scales. When SES are represented in form of a network, the organizational scale is composed of nodes, such as natural components, resource users, civil players, voters, economic actors or regulatory organizations, and of linkages between those nodes, like exchanges or transfers of money, energy, information and strategies.[1]

A network consists of nodes which represent the entities of interest and of edges which embody their interactions. Although networks provide opportunity to study a large variety of systems, among which the socio-ecological ones, their framework does not account for interconnected systems such as the networks of networks (Baggioa et al., 2016). Provided that nodes have different kinds of interactions, creating a layered network, or a multiplex network, where each layer represents a different type of interaction, proves relevant. The network layers are then constituted of links of different types. The field of multilayer networks includes such multiple layers of complexity, as it specifically allows one to differentiate and model the intra-layer and inter-layer connections (Lee et al., 2015; Pilosof et al., 2017).

The key to extracting useful information from these SES models is to address the degree and level of connectivity, which we define as the property of all types of elements

[1] Interest in SES has grown with the awareness that ecosystems evolve with respect to the social organization (Brondizio et al., 2009). Human societies and their institutions are thus central when comes to studying the ecological systems (Halliday and Glaser, 2011). At a global level, an SES can be seen as Earth system where human agents act on patterns of global change (Schellnhuber et al., 2004). At a territorial level, it can be illustrated as a framework for implementing ecological solidarity within the policies of protection of natural areas that revolve within a territory (Mathevet et al., 2016; Frank et al., 2017). At a local level, SES can be used to study family farms, the organization of which is determined not only by the farmers and their economic constraints, but also by the plants and animals (Halliday and Glaser, 2011).

interacting on a network. Connectivity then refers to any form of assemblage, interaction or linkage between human and non-human agents (Nicholls et al., 2016). Its expression can take multiple forms, of which economic (Stromquist, 2002; Wenz and Levermann, 2016), social (Miritello, 2013), environmental (Noss, 1987; Crooks and Sanjayan, 2006; Moritz et al., 2013; Dragicevic et al., 2017), technological (Webb, 2007) and organizational (Tillquist, 2002; Unhelkar, 2009). Besides, those interactions can be between and within economic, social and environmental systems. The performance of complex systems and networks such as SES depends on their ability to maintain the topological structures through the vertex connectivity (Frank and Frisch, 1970).

The analysis of SES sustainability has been mostly conducted through the idea of resilience (Gonzalès and Parrott, 2012). A system is considered to be resilient when it adapts to external perturbations while continuing to function, be it at the expense of changes in the configuration. Public policies toward resilience must then overcome budget limitations, address trade-offs, be acceptable to many competing interests, and overcome barriers in the structure of existing institutions (Carpenter et al., 2012). With a view to reaching resilience, Dragicevic and Shogren (2017) subjected an SES multiplex network to dynamics of reform through the knock-on effect, such that the spread of reform on a node came from the neighborhood or from the counterparts previously reformed. They found that proportional weighting of all realms constitutive of sustainable development yielded the maximum magnitude of efficiency of the knock-on effect. Nonetheless, for a reform to spread on a multiplex network, the connectivity between the nodes needs to be operational.

Researchers and policymakers recognize that in reality risks exist within an SES system that can work to undercut the connectivity due to some exogenous/endogenous economic or ecological barrier/constraint, which will undermine the goal sustainability. Forces such as global environmental change and globalization have pushed connectivity to such a level (Young et al. 2006; Brondizio et al., 2009), that impacts of connections and disconnections on the governance of interactions need to be fully integrated (Clark, 2000). These risks can be both intra- and inter-layer disconnections. An intra-layer disconnection – within each layer – means the termination of an interaction. Examples of the risk to connectivity include trade that no longer takes place, emergence of social distrust, or a destruction of an ecological corridor. An inter-layer disconnection – between the layers – can be illustrated in the context of absence of an equitable society, pointing to the lack of contribution of economic capital in social development; that of an unbearable environment, where society is unresponsive to the natural environment; and by the absence of a viable economy, such that economic growth is pursued against the environment.

In parallel, small research agenda has been devoted to the economic value of connectivity. Dragicevic et al. (2017) considered the construction of ecological networks in forest environments as the optimal control dynamic graph-theoretic problem. Through shadow prices, they managed to provide an economic value to the network connectivity, which was found to be of aggregated nature. Afterward, Dragicevic (2017) considered the spacetime discounting in order to compute the present value of connectivity at the scale of a graph. Through numerical simulations, he found that securing connectivity was much more sensitive to spatial discounting than to the temporal one, implying that agents valued the safeguard of connections less in time than in space.

Herein we develop a model that reveals the preservation value of maintaining connectivity within an SES system. By means of a multiplex network, built from the layers composing the sustainability Venn diagram, we define two measures of preservation value of inter- and intra-layer connections. Most policymakers and researchers have tended to assume that all elements within an SES are unconditionally connected, which implied that dimensions of sustainability functioned unhindered. We must instead explicitly explore how connectivity is operational. Given explicit threats to connectivity, we show under which conditions connectivity is valuable and should be preserved. This implies that policies aimed at sustainability should focus on explicitly addressing connections and disconnections. Using numerical simulations, our results suggest that the preservation value of the SES topological structure is greatest when we secure the connectivity of inter-layer connections.

After this starting section, we present the graph-theoretic characterization of optimal control in Section 2. Section 3 is devoted to illustrating simulation examples. Section 4 concludes.

2 Model

Consider an undirected and unweighted multiplex network, based on the Euclidean metric of dimension $\mathbb{R}^N$. The network is represented by an undirected graph $\Gamma = \{V, E\}$, which consists of vertices $V = \{1, ..., N\}$ indexed by the node members, where i and j represent two neighboring nodes, and of the set of edges $E = \{(i, j) \in V \times V\}$, which represent the inter-node interactions. Fig. 1 illustrates the SES framework composed of three layers with a graph-theoretic mapping.

The population of nodes is distributed among L_n layers, where $n = 1, 2, 3$. Each layer contains N_n nodes, with $i_n = 1, ..., N_n$, with different intra-layer connectivity. Such a multiplex system is completely specified by the vector of the adjacency matrices of the n layers. Let A^n, for $n = 1, 2, 3$, be the adjacency matrix[2] of L_n with nonnegative elements $(a^n_{ij})_{N\times N}$, for $i_n = 1, ..., N_n$. Consider two nodes to be connected when $(i, j) \in E$ such that $a^n_{ij} = 1$; and $a^n_{ij} = 0$ otherwise. Each node in L_n is connected to its counterparts in $L_{\forall n\setminus\{\cdot\}}$,[3] such that there exists a one-to-one connectivity pattern between the identical nodes of different layers. The set of edges E and graph Γ vary in finite time for $t \in [0, T]$.

[2]The adjacency matrix of an undirected graph is symmetric.

[3]$L_{\forall n\setminus\{\cdot\}}$ should read for all layers of n but the one at stake.

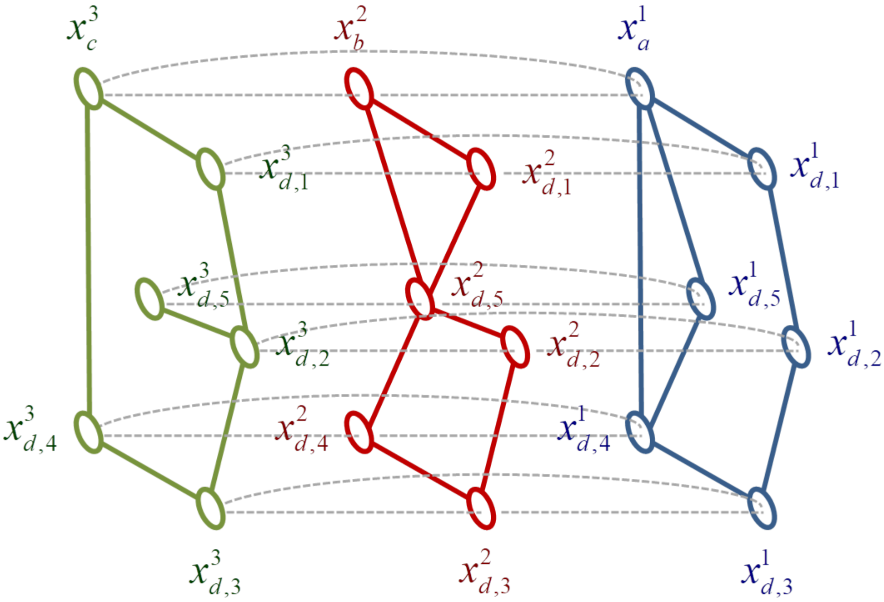

Figure 1: The SES multiplex graph, inspired by the sustainability Venn diagram, is composed of economic (blue), social (red) and environmental (green) layers. Each of them is composed of six connected nodes, that is, $N_1 = \left\{x_a^1(e_a^1), x_{d,1}^1(e_{d,1}^1), x_{d,2}^1(e_{d,2}^1), x_{d,3}^1(e_{d,3}^1), x_{d,4}^1(e_{d,4}^1), x_{d,5}^1(e_{d,5}^1)\right\}$, $N_2 = \left\{x_b^2(e_b^2), x_{d,1}^2(e_{d,1}^2), x_{d,2}^2(e_{d,2}^2), x_{d,3}^2(e_{d,3}^2), x_{d,4}^2(e_{d,4}^2), x_{d,5}^2(e_{d,5}^2)\right\}$ and $N_3 = \left\{x_c^3(e_c^3), x_{d,1}^3(e_{d,1}^3), x_{d,2}^3(e_{d,2}^3), x_{d,3}^3(e_{d,3}^3), x_{d,4}^3(e_{d,4}^3), x_{d,5}^3(e_{d,5}^3)\right\}$. A set of thirty-eight edges, which can be either intra- or inter-layer connections, forms the SES multilayered network. Spatial discount factor δ^l, where $\sum_{l=1}^{10} \delta^l$ in the longest path, weighs up the distance between two nodes.

Now assume the existence of a convex hull of vertices Ω to be an N-simplex, with the Euclidean norm in $\mathbb{R}^N$. Let $x_i^n(e_i^n, t) \in \mathbb{R}^N$, where $n = 1, 2, 3$ and $i_n = 1, ..., N_n$, denote the state of node i_n, characterized by its feature e_i^n at time t.[4] This state encompasses two characteristics: (1) its scalar value, which permits computing its distance from the neighboring nodes; and (2) its nature, like its affiliation to a layer or its similarity with the other nodes.

The set of all possible states of the dynamic system is the configuration space. It is spanned by the stack vector of all the control inputs $x = \left[x_1^{nT}, ..., x_N^{nT}\right]^T$, which denotes the global state vector. The state of each node evolves according to the dynamics which maps control inputs to states through

$$\dot{x} = u_{i_n} \tag{1}$$

where u_{i_n} denotes the control input of node i_n. The latter is selected such that the network evolution is constrained to invariant reachable sets. It drives the network from any initial condition to some arbitrary point in finite time and implies $\dot{x} = 0$ at steady state.

2.1 Distance

Let Λ be the set of nodes, such that the nodes connected to node $i \in \Lambda$ are referred as to subset Λ^i. For $\forall i, j \in \Lambda$, $d_{ij} = |x_i(e_i) - x_j(e_j)|$, and for $\Lambda^i = \{j \in \Lambda : 0 < d_{ij} \leq z\}$, d_{ij} and z respectively stand for the Euclidean distance between nodes, and their respective interaction capacity. To measure utility between two connected nodes, we take into account the capacities in flows, for networks built on large distances are considered to be difficult to arrange. Nodes thus obtain and provide utility u_{ij} from and to other nodes, which can be defined as follows.

Definition 1 *For $\forall i, j \in \Lambda$, $d_{ij} = |x_i(e_i) - x_j(e_j)|$, and for $\Lambda^i = \{j \in \Lambda : 0 < d_{ij} \leq z\}$,*

$$u_{ij} \begin{cases} > 0 & \text{if } \; a_{ij} > 0 \\ = 0 & \text{otherwise} \end{cases}$$

By that, an improperly connected network implies the lack of value creation. Although the existence of nodes could in itself be valued through utility, and their connections defined as sources of positive externalities, we assume that only connections provide utility, knowing that the nodes predate the network construction.

All nodes connected to node i form its utility set $U_i(\Lambda)$. In our case, the network utility set reflects the overall adequate connectivity in the network.

[4] In what follows, the explicit indication of time, such as in $x_i^n(e_i^n, t)$, is not specified unless necessary.

Definition 2 *For $i, j \in \Lambda$, the network utility set of node i is the union of utilities issued from the network intra- and inter-layer connections or $U_i(\Lambda) = \bigcup_{j \in \Lambda} u_{ij}$.*

The network utility can thus be interpreted as the connectivity of the set of relevant nodes separated by distances which satisfy the capacities to interact.

The Euclidean distance becomes irrelevant if the interaction occurs between nodes defined as incompatible. In other words, a node sufficiently close, but endowed with a different feature, cannot provide any utility to the interacting node. For that reason, let us introduce the Mahalanobis distance. The latter measures the dissimilarity between the vectors and accounts for the variance of each variable and the covariance between the variables. As such, smaller distances correspond to interacting nodes that are designated as similar or compatible (Dragicevic et al., 2017).

Following Shaw et al. (2011), consider an Euclidean distance metric parameterized by a positive semidefinite matrix $\Pi = L^T L \equiv S^{-1}$, where $\Pi \in \mathbb{R}^{N \times N}$ and $L \in \mathbb{R}^{N \times N}$.[5] The latter reflects the feature similarity between the nodes, with L a positive semidefinite Laplacian matrix (Godsil and Royle, 2001). It is considered to be network structure preserving if the weighted graph $\Gamma(V, E, \Sigma)$ yields $A^n(\Gamma)$, with A^n, for $n = 1, 2, 3$, the adjacency matrix.

According to the foregoing, the three-dimensional Mahalanobis distance $m(C)_{abc}$ between nodes $a \in \Lambda_1$, $b \in \Lambda_2$ and $c \in \Lambda_3$, issued from the three layers, corresponds to

$$m(C)_{abc} = \left[\left(x_a^1(e_a^1) - \sum_{l=0}^{L} \delta^l \left(x_b^2(e_b^2) + x_c^3(e_c^3) \right) \right)^T S^{-1} \left(x_a^1(e_a^1) - \sum_{l=0}^{L} \delta^l \left(x_b^2(e_b^2) + x_c^3(e_c^3) \right) \right) \right]^{\frac{1}{2}} C \quad (2)$$

where $\sum_{l=0}^{L} \delta^l$ denotes the composite factor of the spatial discounting dependent on the sequence of vertices the distances of which are being measured. When the metric is the identity matrix or $\Pi \equiv S^{-1} = I$, $m(C)_{abc}$ falls back to the standard Euclidean distance between a, b and c. In order to ensure flowing through the network, the nodes specified as compatible shall be linked, which is verified by $S^{-1} \geq 0$. In different words, the Mahalanobis distance metric guarantees that the connection occurs when two nodes display compatible characteristics.

In light of finiteness of resources, nodes that interact build a grid dependent of their opportunity costs.[6] Let scalar C be this economic opportunity cost, from choosing either node from the multiplex graph, computable at the market value.

A network administrator is able to identify the subset of nodes Λ_n, for $n = 1, 2, 3$, evolving on either layer through intra-layer connections, which all have counterparts on other layers. For example, for $n = 1, 2, 3$, we have $\Lambda_{n \setminus \forall n \setminus \{.\}}$ such that

[5]The matrix Π is equivalent to the inverse of the covariance matrix S^{-1}. If two vertices are unconnected they are conditionally independent in the graph (Bell et al., 2000).

[6]The opportunity cost applies to two mutually exclusive options and refers to a benefit that an agent could have received, but gave up, to choose either option.

$$(\Lambda_1 \cup \Lambda_2 \cup \Lambda_3) \setminus (\Lambda_4 \cup \Lambda_5) = \Lambda \text{ and } (\Lambda_1 \cap \Lambda_2 \cap \Lambda_3) \setminus (\Lambda_4 \cap \Lambda_5) = \varnothing \quad (3)$$

The number of nodes in each subset is respectively given by $|\Lambda_1| = N_1$, $|\Lambda_2| = N_2$ and $|\Lambda_3| = N_3$. As for subsets Λ_4 and Λ_5, they correspond to regions delimiting inter-layer connections.

2.2 Dynamics

Following Gustavi et al. (2010), the follower node dynamics is given by the Laplacian-based control strategy (consensus) differential equation, meaning that the state of a node evolves according to the states of the nodes to which it is connected. In detail, the rate of change of a node's state is governed by the sum of states of the neighboring nodes. This property provides evidence for the cascade effects (Dragicevic, 2017).

The dynamics for a node can be written as

$$\dot{x}_a^1(e_a^1) = -Nx_a^1(e_a^1) + \frac{\delta^{L+1} - 1}{\delta - 1} \left[N \left(x_b^2(e_b^2) + x_c^3(e_c^3)\right) - \left(x_d^1(e_d^1) + x_d^2(e_d^2) + x_d^3(e_d^3)\right)\right] \quad (4)$$

where $x_a^1(e_a^1)$ is the state of a node, $x_b^2(e_b^2)$ and $x_c^3(e_c^3)$ are the states of counterparts from the other layers, be it the inter-layer connectivity. As for $x_d^1(e_d^1)$, $x_d^2(e_d^2)$ and $x_d^3(e_d^3)$, they denote the states of nodes connected through intra-layer connectivity. The expression within parentheses is weighted by the spatial discount factor with respect to node i, for $i_n = 1, ..., N_n$. It corresponds to the graph diameter (West, 2000), be it the largest number of nodes which must be traversed in order to travel from one node to another. Its annulment yields the node equilibrium state. It can be formulated through the following lemma.

Lemma 1 *The network equilibrium under consensus dynamics corresponds to the annulment of differential equation $\dot{x}_a^1(e_a^1)$ weighted by the spatial discount factor up to the graph diameter.*

The proof is provided in the appendix.

The following theorem ensues.

Theorem 1 *Given the consensus problem is well-defined in the initial state, the SES equilibrium is at a steady state when the whole is greater than the sum of its parts (i.e., formally, when the marginal variation of the SES barycenter exceeds the marginal variation of a node's utility set).*

The proof is provided in the appendix.

The result shows that, for SES to remain in equilibrium, its center of gravity needs to be greater than the aggregate of states of nodes connecting the layers with respect to the graph diameter or in the longest path. Let us now derive general conditions for the layers to remain connected.

2.3 Connectivity

The connectivity relation $\dot{m}(C)_{abc}$ defines the preservation of the network connectedness. In other words, the network does not disconnect in time. Under the assumptions on differentiability and boundedness of dynamics, the initial connectivity between two nodes remains valid in time if the time derivative of the Mahalanobis distance between them is nonpositive. Thus, the condition for nodes i and j to evolve connected is $\dot{m}(C)_{ij} \leq 0$. When the latter is true, it proves that the convex hull Ω containing the nodes is invariant and, therefore, that the network is Lyapunov-stable (Dragicevic and Sinclair-Desgagné, 2013). The time derivative $\dot{m}(C)_{ij}$ may not be defined when $m(C)_{ij} = 0$, so the squared distance derivative shall be considered instead (Gustavi et al., 2010). It depends on dynamics of nodes i and j and equals

$$\dot{m}(C)^2_{ij} = 2\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b)+x^3_c(e^3_c)\right)\right)^T S^{-1}\left(\dot{x}^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(\dot{x}^2_b(e^2_b)+\dot{x}^3_c(e^3_c)\right)\right)\right] C^2 \quad (5)$$

For arbitrary nodes $a \in \Lambda_1$, $b \in \Lambda_2$ and $c \in \Lambda_3$, the connectivity is defined by

$$\begin{aligned}
\dot{m}(C)^2_{abc} &= 2N\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b)+x^3_c(e^3_c)\right)\right)^T S^{-1}\left[-x^1_a(e^1_a)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right)\right]\right] C^2 \\
&+ 2N\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b)+x^3_c(e^3_c)\right)\right)^T S^{-1}\left[x^2_b(e^2_b)+x^3_c(e^3_c)\right]\right] C^2 \\
&- 6\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b)+x^3_c(e^3_c)\right)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d)+x^2_d(e^2_d)+x^3_d(e^3_d)\right)\right]\right] C^2
\end{aligned} \quad (6)$$

The condition for nodes to evolve connected is $\dot{m}(C)_{ij} \leq 0$. This results in having $\frac{\delta^{L+1}-1}{\delta-1} < \left(\frac{x^2_b(e^2_b)+x^3_c(e^3_c)}{x^1_a(e^1_a)} - 1\right)/2$. Given that $\frac{\delta^{L+1}-1}{\delta-1} \in [0,1]$, the inequality is verified when the states of nodes connected between different layers are equal.

Corollary 1 *Necessary and sufficient condition for arbitrary nodes from different layers to evolve connected in a multilayered graph is that their states be similar.*

The proof is provided in the appendix.

The corollary highlights the egalitarian aspect to be applied to nodes involving on SES, which could be read in conjunction with social justice promoted within sustainable development policies. It also confirms the idea supported in Dragicevic and Shogren (2017) that equal weight should be assigned to all constituents of sustainability.

2.4 Optimal control problem

To guarantee the maintenance of the network connectivity, we apply the optimal control method. It drives the states of the nodes by adjusting the values of control inputs. Following the methodology by Mesbahi and Egerstedt (2010), let us introduce the performance function J, which measures the preservation of the network weighted by the opportunity costs. Because of the constraints relative to the connectivity, it is in form of a standard cost function that is integrated over finite time, such that for all $0 \leq t \leq T$

$$J = \int_0^T \sum m(C)_{abc} \, dt \tag{7}$$

The network administrator decides to sustain the network topology, via control variables $m(C)_{abc}$ in which control is assumed to be the creation of edges associated with the graph (Sengupta and Lafortune, 1992), such that the continuous time optimal control problem can be seen as that of maintaining connectivity, defined over the state variables, which are subject to consensus dynamics, under the constraint of network equilibrium (Lachner et al., 1998). Provided the cost of doing so, as well as its impact on the alternative option, his or her optimal control problem can be formulated as the minimization of the performance function. Put differently

$$\min_{x_a^1(e_a^1),\ x_d^1(e_d^1)} J \tag{8}$$

subject to two first-order dynamic constraints

$$\dot{x}_a^1(e_a^1),\ \dot{x}_d^1(e_d^1) \tag{9}$$

Unlike the standard control law, the problem relates together to the choice of the control vector and the presence of constraints on the state vector. Indeed, the updating of the node state being invariably conducted from the rest of the network, which both includes nodes from the same layer and counterparts from the other layers, we need to look at the first-order necessary optimality conditions of both the control and state components. The states of counterparts from different layers are used as inputs to the network.

The optimal control problem is solved by means of the present value Hamiltonian, discounted in time up to $t = T$, which represents the impact of evolution of $x_a^1(e_a^1)$ and $x_d^1(e_d^1)$ on the network topology. The first-order optimality conditions yield

$$\lambda \;=\; \mu\frac{\delta + 2\delta^{L+1} - 3}{2\delta(1-\delta^{L})} \tag{10}$$

The costate variables, obtained by relaxing the connectivity constraints (Lyon, 1999), are represented by λ and μ. They reveal the shadow prices for keeping the network connected and thus express the network connectivity value: λ for the connectivity between the layers and μ for the connectivity within a layer. The former equality is part of the initial conditions on the choices of costate variables for the system control, such that Theorem 1 holds.

Let $w_0 = \left[x^1_{a_0}(e^1_a)^T, x^2_{b_0}(e^2_b)^T, x^3_{c_0}(e^3_c)^T, x^1_{d_0}(e^1_d)^T, x^2_{d_0}(e^2_d)^T, x^3_{d_0}(e^3_d)^T, \lambda_0^T, \mu_0^T\right]^T$ be the initial network state. In order to control the network, the task consists in fixing λ_0 and μ_0 such that

$$\begin{aligned} x^1_{a_T}(e^1_a) \;&=\; \frac{\delta^{L+1}-1}{\delta-1}\left[x^2_{b_T}(e^2_b) + x^3_{c_T}(e^3_c)\right] \\ &-\; \frac{\delta^{L+1}-1}{N(\delta-1)}\left[x^1_{d_T}(e^1_d) + x^2_{d_T}(e^2_d) + x^3_{d_T}(e^3_d)\right] \end{aligned} \tag{11}$$

where the choices of λ_0 and μ_0 are constrained by (10).

Lemma 2 *The preservation of SES inter- and intra-layer connections is subject to imperfect strategic substitutability.*

The proof is provided in the appendix.

Thereby, the willingness-to-pay for maintaining the connectivity between counterparts from different layers is equal to the willingness-to-pay for preserving the connectivity between nodes within a layer weighted by spatial discounting up to the graph diameter. Otherwise, controlling the network is non-optimal.

By letting $w = \left[x^1_a(e^1_a)^T, x^2_b(e^2_b)^T, x^3_c(e^3_c)^T, x^1_d(e^1_d)^T, x^2_d(e^2_d)^T, x^3_d(e^3_d)^T, \lambda^T, \mu^T\right]^T$ reflect the network state, where the values of shadow prices comply with (10), the system control is obtained through the following Hamiltonian system.

$$\dot{w} = Pw \tag{12}$$

where $P =$

$$\begin{bmatrix} 0 & 0 & \frac{-1}{2C^2S^{-1}e^{-\delta t}\left(1+2\frac{\delta^{L+1}-1}{\delta-1}\right)} & 0 \\ 0 & 0 & \frac{-1}{6C^2S^{-1}e^{-\delta t}} & \frac{-1}{6C^2S^{-1}e^{-\delta t}} \\ -2N\left[S^{-1}C^2e^{-\delta t} + \frac{\delta^{L+1}-1}{\delta-1}\right] & -3\frac{\delta^{L+1}-1}{\delta-1} & N & 0 \\ 0 & 0 & \frac{\delta^{L+1}-1}{\delta-1} & \frac{\delta^{L+1}-1}{\delta-1} \end{bmatrix}$$

Theorem 2 *The SES multiplex state w evolves optimally according to coordinates P.*

The proof is provided in the appendix.

In order to control the network, the costate variables must not invalidate Theorem 2.

3 Simulations

Based on the properties and conditions previously obtained, the aim of this section is to discuss, through simulations, the conditions that guarantee the preservation of SES connectivity in time. For the sake of simplicity, consider a reduced version of the SES multiplex network. Let $n = 1, 2, 3$ represent the three SES layers and $i = 1, ..., 9$ the nodes composing the multilayered network. Each layer is then formed of three nodes. The vector of states, such that $x(0) = [10 \quad 9 \quad 8 \quad 5 \quad 4 \quad 4 \quad 3 \quad 5 \quad 2]$, issued from the subsets of economic, social and environmental nodes, that is $N_1 = \left\{x_a^1(e_a^1), x_{d,1}^1(e_{d,1}^1), x_{d,2}^1(e_{d,2}^1)\right\}$, $N_2 = \left\{x_b^2(e_b^2), x_{d,1}^2(e_{d,1}^2), x_{d,2}^2(e_{d,2}^2)\right\}$ and $N_3 = \left\{x_c^3(e_c^3), x_{d,1}^3(e_{d,1}^3), x_{d,2}^3(e_{d,2}^3)\right\}$, is converted into a square-form distance matrix. We are interested in the behavior of the triad of nodes $(x_a^1(e_a^1), x_b^2(e_b^2), x_c^3(e_c^3))$ issued from the sustainability pillars, which provide connectivity between the three layers, and in the behavior of the remaining nodes, the union of which forms the intra-layer connections. Now consider the following covariance matrix.

	e_a^1	e_b^2	e_c^3	$e_{d,1}^1$	$e_{d,2}^1$	$e_{d,1}^2$	$e_{d,2}^2$	$e_{d,1}^3$	$e_{d,2}^3$
e_a^1	1.00	0.75	0.50	0.90	0.80	0.70	0.75	0.60	0.50
e_b^2	0.75	1.00	0.60	0.65	0.40	0.45	0.80	0.55	0.30
e_c^3	0.50	0.60	0.75	0.20	0.35	0.55	0.45	0.90	0.85
$e_{d,1}^1$	0.90	0.65	0.20	1.00	0.90	0.25	0.40	0.10	0.15
$e_{d,2}^1$	0.80	0.40	0.35	0.90	0.80	0.45	0.65	0.25	0.30
$e_{d,1}^2$	0.70	0.45	0.55	0.25	0.45	1.00	0.85	0.55	0.50
$e_{d,2}^2$	0.75	0.80	0.45	0.40	0.65	0.85	1.00	0.95	0.70
$e_{d,1}^3$	0.60	0.55	0.90	0.10	0.25	0.55	0.95	0.85	0.95
$e_{d,2}^3$	0.50	0.30	0.85	0.15	0.30	0.50	0.70	0.95	1.00

The Laplacian dynamics applied to the nodes yields the following evolution of Mahalanobis coordinates. We suppose that the multilayered network is initially in equilibrium and connected at $t = 0$. In order to see whether the SES connectivity is in jeopardy, let us expose the nodes to the Laplacian laws of motion. The evolution of coordinates is shown in Fig. 2. We can see that distances between the nodes within and between layers enlarge along the timeline. The SES multiplex network is no longer in equilibrium at $t = 50$, in that the average gap between the triad states is increasing in time, going from 1.33 at $t = 0$, with 1.90 at $t = 25$, to 3.67 at $t = 50$. The absence of maintenance of the opening equilibrium goes against the constraints from Lemma 1 or Corollary 1. At the multiplex graph level, the nodes are likely to disconnect which can jeopardize their ability to maintain the convex hull unchanged. In detail, we have $\frac{1}{9}(6 \times 0.03) = 0.02 < (2 \times 0.02) = 0.04$, such that the marginal variation of the SES barycenter is less than the marginal variation of node a's utility set. Therefore, the network equilibrium is not at a steady state. Theorem 1 is thus valid but unverified along the time path in the simulated example.

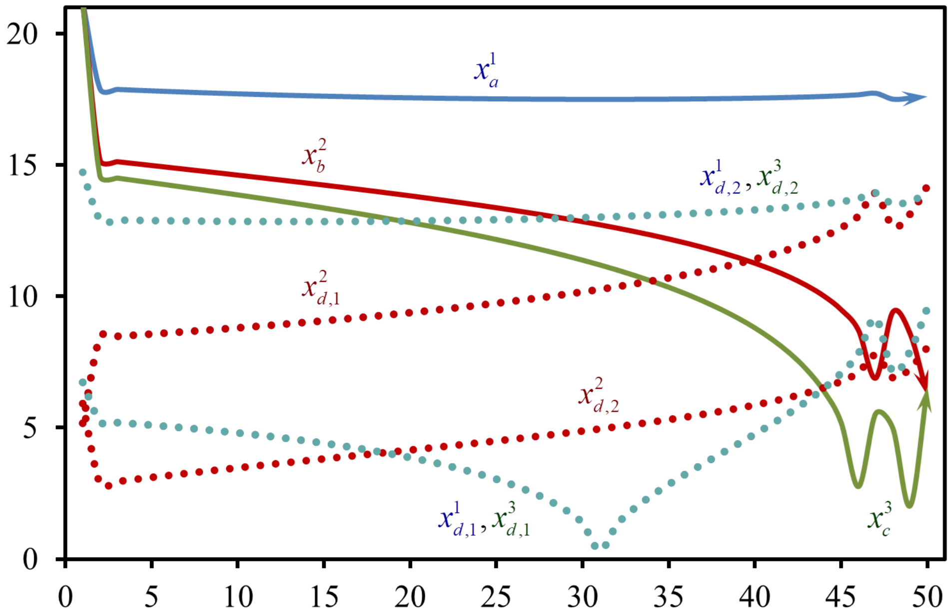

Figure 2: Mahalanobis coordinates (ordinates), spatially discounted at $\delta^l = 0.02$, as functions of time (abscissa) of three inter-layer connected nodes $\left(x_a^1(e_a^1), x_b^2(e_b^2), x_c^3(e_c^3)\right)$, coupled with six additional nodes $\left(x_{d,1}^1(e_{d,1}^1), x_{d,2}^1(e_{d,2}^1), x_{d,1}^2(e_{d,1}^2), x_{d,2}^2(e_{d,2}^2), x_{d,1}^3(e_{d,1}^3), x_{d,2}^3(e_{d,2}^3)\right)$ which constitute the intra-layer connections.

Let us first analyze the inter-layer connectivity. As can be observed in the figure, after the initial decrease in the states of the triad at stake, $x_a^1(e_a^1)$ remains stable throughout the timeline, while $x_b^2(e_b^2)$ and $x_c^3(e_c^3)$ experience a steady decrease from $t = 2$ until $t = 45$. After that time step, an oscillation between rise and fall is observed. The interesting feature of what is noted stands at the stabilization of the economic agent at the expense of both social and environmental agents. To take the analysis one step further, the swinging of social and environmental nodes starting at $t = 46$ is reflective, such that the state improvement of the former seems to deteriorate the state of the latter, which can, for example, illustrate the substitutability between labor market improvement and environmental degradation widely documented in the literature (Lawn, 2006). Just as much, triads $\left(x_{d,1}^1(e_{d,1}^1), x_{d,1}^2(e_{d,1}^2), x_{d,1}^3(e_{d,1}^3)\right)$ and $\left(x_{d,2}^1(e_{d,2}^1), x_{d,2}^2(e_{d,2}^2), x_{d,2}^3(e_{d,2}^3)\right)$ show the substitution between the three pillars of sustainability. Nevertheless, unlike in the previous pattern, the economic $\left(x_{d,1}^1(e_{d,1}^1), x_{d,2}^1(e_{d,2}^1)\right)$ and environmental $\left(x_{d,1}^3(e_{d,1}^3), x_{d,2}^3(e_{d,2}^3)\right)$ agents now co-evolve in a compatible manner. The most striking feature coming from the overall examination is that social agents appear to be the adjustment variable in the SES framework. Indeed, their states follow the opposite dynamics of either or both the economic and environmental nodes.

As for the intra-layer connectivity, the broad analysis reveals that a different type of substitutability occurs within the layers. Like in the previous case, the rise of two nodes is realized at the cost of the third one. A more detailed examination shows three different dynamics. The first one is relative to the social triad and illustrates the fall of $x_b^2(e_b^2)$ with respect to $\left(x_{d,1}^2(e_{d,1}^2), x_{d,2}^2(e_{d,2}^2)\right)$. The second one applies to the economic triad and reveals an irregular trend. Thereby, $x_a^1(e_a^1)$ and $x_{d,2}^1(e_{d,2}^1)$ first decline and then stand on flat trajectories, which is not the case of $x_{d,1}^1(e_{d,1}^1)$ that has a v-shaped path – with an inflection at $t = 31$ – before faltering after $t = 45$. The third pattern can be considered as trendless. That is, while $x_{d,2}^3(e_{d,2}^3)$ is almost stable from $t = 2$ until the terminal time step, $x_c^3(e_c^3)$ and $x_{d,2}^3(e_{d,2}^3)$ start to oscillate at $t = 45$ after a monotonous decrease, with a vacillating trajectory of $x_{d,1}^3(e_{d,1}^3)$ explained earlier through that of $x_{d,1}^1(e_{d,1}^1)$.

Result 1 *The Mahalanobis coordinates free from optimal control indicate that the SES multiplex structure is out of equilibrium and at risk of dismantling.*

Next to the analysis of Mahalanobis coordinates, let us now look at the optimal control conditions for different levels of opportunity costs. As said earlier, the optimal control is meant to secure the coordinates of the SES multiplex graph, at a cost evaluated through shadow prices, such that inter- and intra-layer connections are preserved. Two cases are examined, i.e. (1) the preservation of SES through its inter-layer connections (SES(λ)); and (2) the preservation of SES through its intra-layer connections (SES(μ)). Shadow prices represent the costate variables, the equation of which is detailed in (10), computed by solving the optimal control problem. The first case is studied by considering λ to which the Hamiltonian coordinates from (12) are applied. The outcomes correspond to the preservation values of inter ($\lambda_{\lambda(P)}$) and intra-layer ($\mu_{\lambda(P)}$) connections. The second case is studied by considering μ to which the Hamiltonian coordinates are applied as well. The outcomes then correspond to the preservation values of inter ($\lambda_{\mu(P)}$) and intra-layer ($\mu_{\mu(P)}$) connections.

Fig. 3 illustrates the behavior of shadow prices $(\lambda_{\lambda(P)}, \mu_{\lambda(P)})$, obtained with respect to inter-layer control, with rising levels of opportunity costs, such that $C = [0.50, 1 \times 10^6]$. We observe that $\lim_{C\to 0} \{\lambda, \mu\} = \{-\infty, -\infty\}$ and $\lim_{C\to\infty} \{\lambda, \mu\} = \{115, -0.09\}$.

Fig. 4 depicts the behavior of shadow prices $(\lambda_{\mu(P)}, \mu_{\mu(P)})$, obtained with respect to intra-layer control, with rising levels of opportunity costs, such that $C = [0.50, 1 \times 10^6]$. We observe that $\lim_{C\to 0} \{\lambda, \mu\} = \{-\infty, -\infty\}$ and $\lim_{C\to\infty} \{\lambda, \mu\} = \{6.52, -8, 556\}$.

Result 2 *The shadow prices are negatively unbounded at near-zero opportunity costs and positively bounded with significant levels of opportunity costs.*

Therefore, in both cases, null opportunity costs generate high negative preservation values, whereas significant opportunity costs, starting at $C = 10$, produce ceiling values of preservation. Put differently, when the opportunity cost is close to zero, which implies

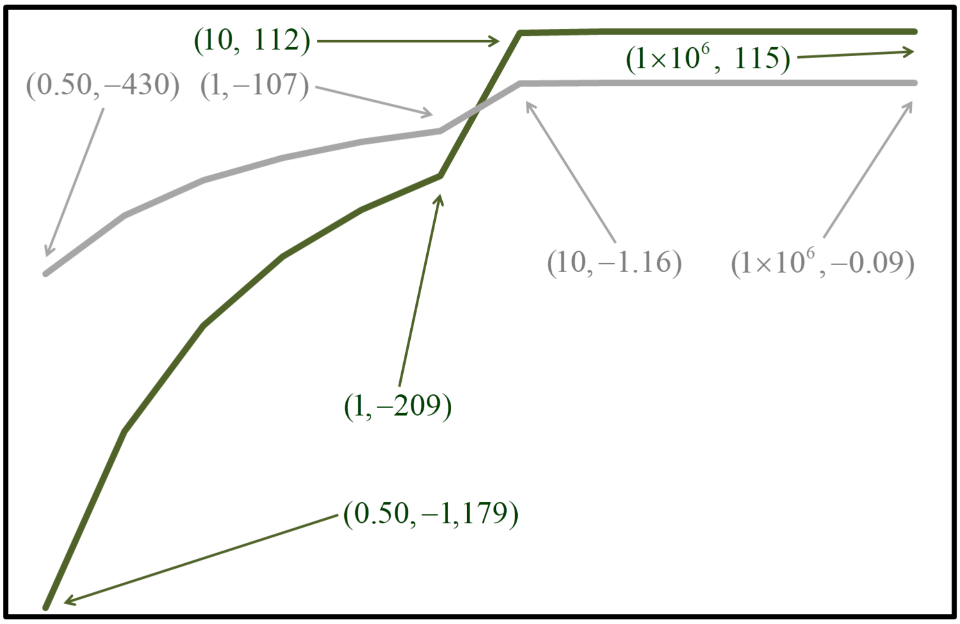

Figure 3: Shadow prices (ordinates) from equal discounting in space and time of $\delta^{l,t} = 0.02$, as functions of opportunity costs (abscissa), for the SES preservation through its inter-layer connections (SES(λ)). The green curve depicts the preservation values of inter-layer connections issued from inter-layer control ($\lambda_{\lambda(P)} \Rightarrow$ SES(λ)). The grey curve represents the preservation values of intra-layer connections issued from inter-layer control ($\mu_{\lambda(P)} \Rightarrow$ SES(λ)).

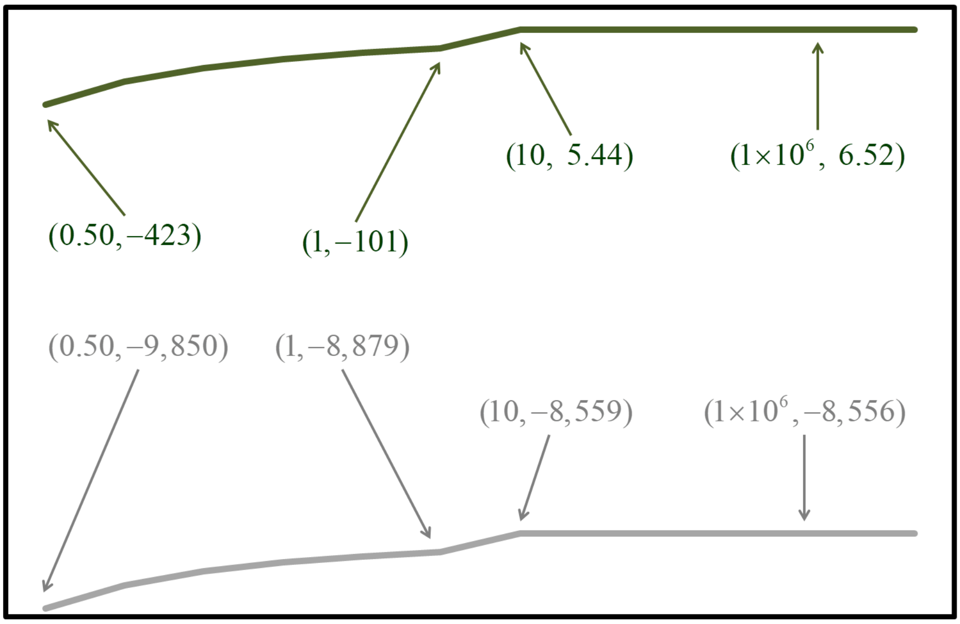

Figure 4: Shadow prices (ordinates) from equal discounting in space and time of $\delta^{l,t} = 0.02$, as functions of opportunity costs (abscissa), for the SES preservation through its intra-layer connections (SES(μ)). The green curve depicts the preservation values of inter-layer connections issued from intra-layer control ($\lambda_{\mu(P)} \Rightarrow$ SES(μ)). The grey curve represents the preservation values of intra-layer connections issued from intra-layer control ($\mu_{\mu(P)} \Rightarrow$ SES(μ)).

indifference toward preserving connectivity between the layers or operating on its own layer without considering the remaining sustainability pillars, the value of connectivity is negative, in that maintaining connectivity between economy, society and environment is a costly undertaking. When the opportunity cost is of significant amount, the value of connectivity is upper-bounded, because the role of SES is to provide utility both within and between layers.

The numerical values show that Lemma 2 is verified for two reasons. First, we see that $\lambda_{\lambda(P)} = -\mu_{\lambda(P)}$ and $\lambda_{\mu(P)} = -\mu_{\mu(P)}$. Thereby, there exists strategic substitutability between preserving connectivity within and between layers. Then, we have $\mid \lambda_{\lambda(P)} \mid > \mid -\mu_{\lambda(P)} \mid$ and $\mid \lambda_{\mu(P)} \mid \gg \mid -\mu_{\mu(P)} \mid$. In that, strategic substitutability is imperfect in either case, and also far from being proportional in the second case.

At a first glance, preserving the SES structure by investing in inter-layer connections seems irrelevant. With respect to the costs to be incurred via $\lambda(P)$, maintaining the SES intact is cost consuming. On the contrary, securing both inter- and intra-layer connections via $\mu(P)$ requires a moderate investment. However, due to imperfect substitution noted above, a closer look on the values of $\lambda_{\lambda(P)}$ and $\mu_{\lambda(P)}$ proves the exact opposite. Simply because preserving the SES structure from within the layers appears less onerous, risking a low-cost intra-layer control of 6.52, for a possible damage of $-8,556$ on intra-layer connectivity, against a high-cost inter-layer control of 115, for a damage of only -0.09 on intra-layer connectivity, can be considered as economically unsound. Not only because the sacrificed connections would then need to be rehabilitated in the long-run, but also in the wake of compensatory damages that would need to be paid.[7] In fact, the shadow price is the change in the optimal control solution obtained by relaxing the constraint of connectivity. Economically speaking, it is the maximum price one is willing to pay to maintain the network connected for an additional unit of time. If a negative value is the willingness-to-accept a monetary compensation for letting the SES connectivity deteriorate, one would only ask for an offset of 0.09 in case of investment in inter-layer connections, and of $8,556$ in case of investment in intra-layer connections.

Result 3 *Preserving the SES structure through optimal control is more efficient by securing connectivity of inter-layer connections than of intra-layer connections, because substitutability is more acute in case of intra-layer control.*

Furthermore, if one considers the opportunity costs to be negligible, one could even ask for infinite compensation demanded. Indeed, due to broken links between the layers, one could end up disconnecting from the overall network while mostly evolving with respect to the nodes from its layer neighborhood. This result emphasizes the importance

[7] In a study conducted by OECD (2018), the authors point out that the CATNAT compensation scheme, which was conceived to encourage flood prevention rather than indemnity for damages, simply because the former is less expensive for all the stakeholders, has not been successful to date. When read in conjunction with our results, their conclusions bring to light that the tradeoff between willingness-to-pay (as the cost of prevention) and willingness-to-accept (as the monetary payout) is of utmost importance.

of connections between economy, society and environment that might lead to the collapse of SES simply because one distances oneself from the inter-layer connectivity. This also ties up with the idea that SES are linked systems of people and nature, where economic agents are not apart from the nature (Berkes et al., 2003)

4 Conclusion

By considering the SES system as a multiplex network composed of layers borrowed from the sustainability Venn diagram, we uncovered its properties of equilibrium and performance, the latter being defined as the ability to maintain its topological structure through the vertex connectivity. We could think of the relevance to preserve the SES structure through its inter-layer connections as our main result. The latter ties up with a previous finding, obtained through the resolution of an evolutionary variational inequality, that the stability of compartmentalized networks depends on the preservation of between-subnetwork coupling (Dragicevic, 2016), should the SES multilayered network be seen as a networked system divided into compartments.

A geographically-based cost structure to forming links was presented in Jackson and Rogers (2005). Their structure captured heterogeneity in link costs, where agents grouped on an island bore a lower cost of forming connections within that island and bore a higher cost across the islands. Accordingly, the links within an island represent the intra-layer connectivity, while the links across the islands represent the inter-layer connectivity. Given the benefits of indirect connections, the authors showed that this type of cost structure generated the structure of a small-world network, with higher rates of connections on a local scale, such that the neighbors of any given node were likely to be the neighbors of each other. Through the optimal control method, this work extends their findings in that (1) Result 3 reveals the presence of characteristics of a multiplex small world network (Agarwal et al., 2016); (2) a higher (shadow) cost is meant instead to avoid severing links of any kind in the multilayered network.

The alternative representation of sustainability in form of concentric circles (Mitchell, 2000), in which the economic area is embedded within the social area, which is within the natural environment, is probably the most appropriate to address the challenges faced by the SES system. In the absence of such reasoning at the global scale, working collectively on the safeguard of connections that link economy, society and environment should be the second-best blueprint on which to engage in the medium-term.

The future avenues of research can be classified in two categories. The first category is about the study of weighted socio-ecological multiplex systems, through the analysis of an aggregated topological adjacency matrix, by discriminating between the strength of a node over its neighbors and/or counterparts. That way, we could monitor the dynamics subject to an ex-ante layer discrimination. The second category concerns the graph-theoretic characterization of stochastic controllability, by means of stochastic optimal control, all the more so due to the conjunction of different types of uncertainty (Hambusch, 2008), such as climate change (Joyce et al., 2006), that weigh on SES systems.

As previously stated in other works, this one also has to be considered as exploratory, inasmuch as graph theory is a schematized representation of network patterns. Therefore, complementary works should be conducted in order to endorse or to disapprove the subject matter designed and discussed in this paper.

Acknowledgments

Appendix

2 Model

2.2 Dynamics

Equation (4)

$$
\begin{aligned}
\dot{x}_a^1(e_a^1) &= -\sum_{d\in\Lambda^1}\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_d^1(e_d^1)\right)-\sum_{d\in\Lambda^2}\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_d^2(e_d^2)\right)-\sum_{d\in\Lambda^3}\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_d^3(e_d^3)\right)\\
&+ \sum_{d\in\Lambda^1}\sum_{l=0}^{L}\delta^l\left(x_b^2(e_b^2)-x_d^1(e_d^1)\right)+\sum_{d\in\Lambda^2}\sum_{l=0}^{L}\delta^l\left(x_b^2(e_b^2)-x_d^2(e_d^2)\right)+\sum_{d\in\Lambda^3}\sum_{l=0}^{L}\delta^l\left(x_b^2(e_b^2)-x_d^3(e_d^3)\right)\\
&+ \sum_{d\in\Lambda^1}\sum_{l=0}^{L}\delta^l\left(x_c^3(e_c^3)-x_d^1(e_d^1)\right)+\sum_{d\in\Lambda^2}\sum_{l=0}^{L}\delta^l\left(x_c^3(e_c^3)-x_d^2(e_d^2)\right)+\sum_{d\in\Lambda^3}\sum_{l=0}^{L}\delta^l\left(x_c^3(e_c^3)-x_d^3(e_d^3)\right)\\
&= -N_1x_a^1(e_a^1)-N_2x_a^1(e_a^1)-N_3x_a^1(e_a^1)\\
&+ N_1\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)+N_2\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)+N_3\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)\\
&+ N_1\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)+N_2\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)+N_3\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)\\
&- \sum_{d\in\Lambda^1}\sum_{l=0}^{L}\delta^l x_d^1(e_d^1)-\sum_{d\in\Lambda^2}\sum_{l=0}^{L}\delta^l x_d^2(e_d^2)-\sum_{d\in\Lambda^3}\sum_{l=0}^{L}\delta^l x_d^3(e_d^3)\\
&= -N_1\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)-\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)\right)\\
&- N_2\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)-\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)\right)\\
&- N_3\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)-\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)\right)\\
&- \sum_{d\in\Lambda^1}\sum_{l=0}^{L}\delta^l x_d^1(e_d^1)-\sum_{d\in\Lambda^2}\sum_{l=0}^{L}\delta^l x_d^2(e_d^2)-\sum_{d\in\Lambda^3}\sum_{l=0}^{L}\delta^l x_d^3(e_d^3)\\
&= -N\left(x_a^1(e_a^1)-\sum_{l=0}^{L}\delta^l x_b^2(e_b^2)-\sum_{l=0}^{L}\delta^l x_c^3(e_c^3)\right)-\sum_{l=0}^{L}\delta^l\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\\
&= -Nx_a^1(e_a^1)+\frac{\delta^{L+1}-1}{\delta-1}\left[N\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)-\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right]
\end{aligned}
$$

Proof of Lemma 1. Let regions Λ_1, Λ_2 and Λ_3 be the subsets of Λ such that $(x_a^1(e_a^1), x_d^1(e_d^1)) \in \Lambda_1$, $(x_b^2(e_b^2), x_d^2(e_d^2)) \in \Lambda_2$ and $(x_c^3(e_c^3), x_d^3(e_c^3)) \in \Lambda_3$ through the intra-layer connections. In parallel, the inter-layer connections involve the existence of regions

Λ_4 and Λ_5 which are also defined to be the subsets of Λ, such that $(x_a^1(e_a^1), x_b^2(e_b^2), x_c^3(e_c^3)) \in \Lambda_4$ and $(x_d^1(e_d^1)), x_d^2(e_d^2)), x_d^3(e_c^3)) \in \Lambda_5$. A convex hull of Λ exists if $(\Lambda_1 \cup \Lambda_2) \setminus (\Lambda_4 \cup \Lambda_5)$, $(\Lambda_2 \cup \Lambda_3) \setminus (\Lambda_4 \cup \Lambda_5)$ and $(\Lambda_1 \cup \Lambda_3) \setminus (\Lambda_4 \cup \Lambda_5)$ on the side of intra-layer connectivity, and if $(\Lambda_1 \cap \Lambda_4) \setminus (\Lambda_1 \cup \Lambda_5)$, $(\Lambda_2 \cap \Lambda_4) \setminus (\Lambda_2 \cup \Lambda_5)$ and $(\Lambda_3 \cap \Lambda_4) \setminus (\Lambda_3 \cup \Lambda_5)$ on the side of inter-layer connectivity. We denote $\Omega \times [0, T] = \Lambda \subseteq \mathbb{R}^N$ such a convex hull. The consensus problem is well-defined when the nodes belong to the invariant set $\Omega(0)$. Therefore, $\{(x_a^1(e_a^1), 0), (x_d^1(e_d^1), 0), (x_b^2(e_b^2), 0), (x_d^2(e_d^2), 0), (x_c^3(e_c^3), 0), (x_d^3(e_d^3), 0)\} \in \Omega(0)$, such that the initial states allow for a proper connection or $\underset{j \in \mathbb{R}^N}{\cup} u_{ij}(0) > 0$, where $(i, j) \in \Lambda$ and $(i, j) \in E$. Given that the control input compels the network evolution to invariant sets, there exists a fixed point at steady state (Shakarian et al., 2012), and the consensus equilibrium is unique. Accordingly,

$$
\begin{aligned}
0 &= -N x_a^1(e_a^1) + \frac{\delta^{L+1} - 1}{\delta - 1} \left[N \left(x_b^2(e_b^2) + x_c^3(e_c^3) \right) - \left(x_d^1(e_d^1) + x_d^2(e_d^2) + x_d^3(e_d^3) \right) \right] \\
N x_a^1(e_a^1) &= \frac{\delta^{L+1} - 1}{\delta - 1} \left[N \left(x_b^2(e_b^2) + x_c^3(e_c^3) \right) - \left(x_d^1(e_d^1) + x_d^2(e_d^2) + x_d^3(e_d^3) \right) \right] \\
x_a^1(e_a^1) &= \frac{\delta^{L+1} - 1}{N(\delta - 1)} \left[N \left(x_b^2(e_b^2) + x_c^3(e_c^3) \right) - \left(x_d^1(e_d^1) + x_d^2(e_d^2) + x_d^3(e_d^3) \right) \right] \\
x_a^1(e_a^1) &= \frac{\delta^{L+1} - 1}{\delta - 1} \left[x_b^2(e_b^2) + x_c^3(e_c^3) \right] - \frac{\delta^{L+1} - 1}{N(\delta - 1)} \left[x_d^1(e_d^1) + x_d^2(e_d^2) + x_d^3(e_d^3) \right]
\end{aligned}
$$

The state of node on a layer converges to a value equal to the difference between the aggregate of the states of nodes on other layers to which it is connected and the aggregate of barycenters of the three layers. ■

Proof of Theorem 1. The proof follows from Lemma 1. Let $\dot{x}_a^1(e_a^1) = f(x_a^1(e_a^1))$ be an equilibrium when $f(x_a^1(e_a^1)) = 0$. Differentiation yields $f'(x_a^1(e_a^1)) < 0$ when

$$
x_a^{1'}(e_a^1) > \frac{\delta^{L+1} - 1}{N(\delta - 1)} \left[x_d^{1'}(e_d^1) + x_d^{2'}(e_d^2) + x_d^{3'}(e_d^3) \right] - \frac{\delta^{L+1} - 1}{\delta - 1} \left[x_b^{2'}(e_b^2) + x_c^{3'}(e_c^3) \right] \qquad (13)
$$

where $x_i^{n'}(e_i^n)$ represents the marginal variation of node i's state from layer n. We have $x_a^{1'}(e_a^1) > 0$ when $\frac{\delta^{L+1}-1}{N(\delta-1)} \left[x_d^{1'}(e_d^1) + x_d^{2'}(e_d^2) + x_d^{3'}(e_d^3) \right] > \frac{\delta^{L+1}-1}{\delta-1} \left[x_b^{2'}(e_b^2) + x_c^{3'}(e_c^3) \right]$. ■

2.3 Connectivity

Equation (6)

$$
\begin{aligned}
\dot{m}(C)^2_{abc} &= 2\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left(\dot{x}^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(\dot{x}^2_b(e^2_b) + \dot{x}^3_c(e^3_c)\right)\right)\right]C^2 \\
&= 2\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[-Nx^1_a(e^1_a) + Nx^2_b(e^2_b) + Nx^3_c(e^3_c)\right]\right]C^2 \\
&+ 2\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[N\frac{\delta^{L+1}-1}{\delta-1}\left(-2x^1_a(e^1_a)\right)\right]\right]C^2 \\
&- 6\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right]\right]C^2 \\
&= 2N\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[-x^1_a(e^1_a)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right)\right]\right]C^2 \\
&+ 2N\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[x^2_b(e^2_b) + x^3_c(e^3_c)\right]\right]C^2 \\
&- 6\left[\left(x^1_a(e^1_a) - \sum_{l=0}^{L}\delta^l\left(x^2_b(e^2_b) + x^3_c(e^3_c)\right)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right]\right]C^2
\end{aligned}
$$

Proof of Corollary 1.

$$
\begin{aligned}
2N\left[-x^1_a(e^1_a)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right)\right]C^2 + 2N\left[x^2_b(e^2_b) + x^3_c(e^3_c)\right]C^2 & \\
-6\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right]C^2 &\leq 0 \\
\Leftrightarrow N \leq \frac{6\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right]C^2}{2\left[-x^1_a(e^1_a)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right) + x^2_b(e^2_b) + x^3_c(e^3_c)\right]C^2} & \\
\Leftrightarrow N \leq -\frac{3\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right]}{x^1_a(e^1_a)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right) + x^2_b(e^2_b) + x^3_c(e^3_c)} &
\end{aligned}
$$

Provided that $N > 0$ and $-3\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x^1_d(e^1_d) + x^2_d(e^2_d) + x^3_d(e^3_d)\right)\right] < 0$, we have

$$
\begin{aligned}
-\frac{3\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right]}{x_a^1(e_a^1)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right)+x_b^2(e_b^2)+x_c^3(e_c^3)} &> 0 \\
\Leftrightarrow x_a^1(e_a^1)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right)+x_b^2(e_b^2)+x_c^3(e_c^3) &< 0 \\
\Leftrightarrow x_a^1(e_a^1)\left(2\frac{\delta^{L+1}-1}{\delta-1}+1\right) &< x_b^2(e_b^2)+x_c^3(e_c^3) \\
\Leftrightarrow \frac{\delta^{L+1}-1}{\delta-1} &< \frac{\frac{x_b^2(e_b^2)+x_c^3(e_c^3)}{x_a^1(e_a^1)}-1}{2}
\end{aligned}
$$

■

2.4 Optimal control problem

The Hamiltonian corresponds to

$$
H=\begin{cases}
2C^2\left(x_a^1(e_a^1)-\frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)\right)^T S^{-1}\left[-Nx_a^1(e_a^1)+Nx_b^2(e_b^2)+Nx_c^3(e_c^3)\right]e^{-\delta t} \\
+2C^2\left(x_a^1(e_a^1)-\frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)\right)^T S^{-1}\left[-2N\frac{\delta^{L+1}-1}{\delta-1}x_a^1(e_a^1)\right]e^{-\delta t} \\
-6C^2\left(x_a^1(e_a^1)-\frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right]e^{-\delta t} \\
+\lambda^T\left[-Nx_a^1(e_a^1)+\frac{\delta^{L+1}-1}{\delta-1}\left[N\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)-\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right]\right] \\
+\mu^T\left[-\frac{\delta^{L+1}-1}{\delta-1}\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right]
\end{cases}
$$

The first-order optimality conditions are

$$
\begin{aligned}
\frac{\partial H}{\partial x_a^1(e_a^1)} &= 2C^2\left(x_a^1(e_a^1)-\frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)\right)^T S^{-1}\left[-N-2N\frac{\delta^{L+1}-1}{\delta-1}\right]e^{-\delta t} \\
&- \lambda^T[N]=0
\end{aligned}
$$

from which we obtain

$$
x_a^1(e_a^1)=\frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)-\frac{\lambda}{2C^2S^{-1}e^{-\delta t}\left(1+2\frac{\delta^{L+1}-1}{\delta-1}\right)}
$$

And

$$
\begin{aligned}
\frac{\partial H}{\partial x_d^1(e_d^1)} &= -6C^2\left(x_a^1(e_a^1) - \frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right)\right)^T S^{-1}\left[\frac{\delta^{L+1}-1}{\delta-1}\right]e^{-\delta t} \\
&+ \lambda^T\left[-\frac{\delta^{L+1}-1}{\delta-1}\right] - \mu^T\left[\frac{\delta^{L+1}-1}{\delta-1}\right] = 0
\end{aligned}
$$

which yields

$$
x_a^1(e_a^1) = \frac{\delta^{L+1}-1}{\delta-1}\left(x_b^2(e_b^2)+x_c^3(e_c^3)\right) - \frac{\lambda+\mu}{6C^2S^{-1}e^{-\delta t}}
$$

The boundary conditions are as follows

$$
\begin{aligned}
\dot{\lambda} &= -\left[\frac{\partial H}{\partial x_a^1(e_a^1)}\right]^T \\
&= 2C^2S^{-1}e^{-\delta t}\left[N\left(x_a^1(e_a^1)-x_b^2(e_b^2)-x_c^3(e_c^3)\right)\right] \\
&+ 2C^2S^{-1}e^{-\delta t}\left[\frac{\delta^{L+1}-1}{\delta-1}\left(2Nx_a^1(e_a^1)+3\left(x_d^1(e_d^1)+x_d^2(e_d^2)+x_d^3(e_d^3)\right)\right)\right] + \lambda N
\end{aligned}
$$

And

$$
\begin{aligned}
\dot{\mu} &= -\left[\frac{\partial H}{\partial x_d^1(e_d^1)}\right]^T \\
&= \frac{\delta^{L+1}-1}{\delta-1}(\lambda+\mu)
\end{aligned}
$$

Proof of Lemma 2. From the optimality conditions, we obtain

$$
\begin{aligned}
-\frac{\lambda}{2C^2S^{-1}e^{-\delta t}\left(1+2\frac{\delta^{L+1}-1}{\delta-1}\right)} &= -\frac{\lambda+\mu}{6C^2S^{-1}e^{-\delta t}} \\
\Leftrightarrow \lambda &= \mu\frac{\delta+2\delta^{L+1}-3}{2\delta(1-\delta^L)}
\end{aligned}
$$

We have $\frac{\delta+2\delta^{L+1}-3}{2\delta(1-\delta^L)} < 0$ for $\delta \in (0,1)$, the expression being incalculable when $\delta = \{0,1\}$, such that we are in presence of imperfect strategic substitutes. ■

Proof of Theorem 2. By Theorem 1, we know that the network is at equilibrium. We have determined the first-order necessary optimality conditions to set up the Hamiltonian coordinates. All the necessary and sufficient conditions are met. ■

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