Reconciling Hotelling Resource Models with Hotelling’s Accounting Method

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Abstract. A methodology of accounting is developed that finds that the Hamiltonian is a measure of income in an autonomous problem but that in a non-autonomous problem income should include capital gains. Four canonical models in exhaustible-resource economics are studied. The Hotelling model of a competitive resource is a non-autonomous problem. Capital gains should be included in the gains of resource producers. The corresponding planner’s problem may be autonomous, however. The change in the problem changes the accounting for income in the new numeraire, utility. Lewis Gray’s problem for a competitive firm operating with u-shaped average cost is interpreted as both an autonomous and a non-autonomous problem. Finally, the problem with costs dependent on depletion or remaining reserves (the *stock effect*) is discussed. Accounting must apply to non-optimal resource allocation outcomes, and it is likely that there are sources of non-autonomy when a problem is not optimal.

Key words: exhaustible resources, accounting, income, non-autonomy

# Introduction

Consider the equilibrium for a simple Hotelling resource: a homogeneous stock *S*(*t*), known and available at time *t*, is depleted over time by extracting a quantity *q*(*t*) = -*dS*(*t*)/*dt* in order to provide a flow of net benefits *f*(*q*(*t*),*t*). This flow is discounted at the force of interest *r*(*t*) to form the (net present) value, *V*. In general, the value is a function of the stock *S*(*t*), the starting time *t*, the time of abandonment *T* and the final stock *S*(*T*) (Seierstad and Sydsaeter 1987: 211). Because of the homogeneity of the resource, one normally finds that *S*(*T*) = 0. Since *T* may not be of great interest, one can write *T* = ∞ and allow that the stock may be fully depleted as of some finite date. The depletion of the stock is defined to be *q* = -*dS/dt*, expressed in physical units of the resource per unit time. In general, the problem may be (time) non-autonomous, so that some variables depend explicitly on calendar time; in this case, time is an argument of the functions *f* and *r*. In compact notation, value can be written

. (1)

The flow of benefits may or may not maximize value. In general, the flow and the value are determined by what Dasgupta and Mäler (2000) call a resource-allocation mechanism (RAM), which allocates the stock to different times according to rules and conventions that remain unspecified.

# Some Accounting Fundamentals

Differentiation of equation (1) yields that

 . (2)

Equation (2) is a basic equation for accounting for outcomes in the model. The flow *f*(*q,t*) may not be evaluated at market or even observable prices: it may, for example, be utility or utility net of some cost (Aronsson, Löfgren and Backlund 2004). Accounting for stocks and flows uses the shadow values that apply to the RAM. Prices are expressed in the numeraire assumed in the model. There is no inflation.

Let on the path determined by the RAM. Even though the RAM may not be to optimize value, a (current-value) Hamiltonian function can be defined:

. (3)

The Hamiltonian is traditionally considered to be the net product (cf Weitzman 2003): It is the sum of the net benefits and the net investments evaluated at the shadow values of the RAM. The term *λq* is interpreted as the *value* of the depletion *q*.

Equivalent results can be obtained using the fundamental equation of dynamic programming under the RAM. Over a small time interval *dt*,

. (4)

Simplifying to first order yields that, as in equation (2),

. (5)

Interest on the capital value of the resource, *rV*, is the total return to the stock, i.e., what would have been earned if *V*(*S,t*) had been realized in the market and loaned out for a “period” *dt*.

There is a symmetry among the entries in equation (5); none is more important than another. As easy to write as equation (2) is

$rV=f\left(q,t\right)+\frac{dV}{dt}=f\left(q,t\right)+\frac{∂V}{∂S}\dot{S}+\frac{∂V}{∂t}=f\left(q,t\right)-λq+\frac{∂V}{∂t}=H\left(q,S,t,λ\right)+\frac{∂V}{∂t}$ . (6)

Equation (6) expresses the no-arbitrage condition, and hence how the resource is integrated into the capital market of the wider economy: the total return on the stock is equal to the sum of the dividend, *f*(*q,t*) – *λq = H*, and the capital gain, *∂V*/*∂t*.

These results hold for accounting for extraction to obtain a flow *f*(*q,t*), however defined, when value is expressed as an integral of the discounted flow. Because accounting must apply to any RAM, not just an optimal RAM, they are a foundation for accounting in the Hotelling model.

Two methods of accounting can be discerned.

1. In the traditional method, the current-value Hamiltonian is considered to be the net product. In equation (3), net product is the benefit, *f*(*q,t*), net of the value of depletion, *λq*; the latter is defined to be depreciation or the value of net investment. Net income is equal to net product by an accounting identity. If the capital gain, *∂V*/*∂t*, is not equal to zero, however (if the problem is non-autonomous), the integral of *λq* (undiscounted) is *not* equal to the value of the program. Therefore, it does not satisfy the main property of depreciation, namely, that over time depreciation “adds up” (integrates) to the initial value (Cairns 2018):

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1. In another method based on equation (6), interest on value, *rV*, can be interpreted as net income. The interpretation is consistent with equilibrium in capital markets. There is a direct relationship between income and capital value, *V*. Depreciation is given by , the value of depletion net of the capital gain or loss. This expression does “add up” to the initial value: on the assumption that ,

. It is the depreciation described by Hotelling (1925) and also by Samuelson (1937). Net product is the sum of the Hamiltonian (the benefit net of depletion) and the capital gain or loss,  (cf. equation (6)).

According to the development leading up to equation (6), the capital gain is as much part of the decision making of the extracting firm as the other two terms, and is as “productive” as the other terms. Let the *rate of depreciation* be defined to be *δ* = − (1/*V*)(*dV*/*dt*). Then *f*(*q,t*) = (*r* + *δ*)*V*. The flow of net benefits can be interpreted as the Jorgensonian user cost of the activity of resource extraction (Jorgenson 1963). Equations (1) through (6) suggest that the capital gain should be part of the accounting for the decisions of the firm. Hicksian income no. 1 under the RAM is that level of net benefit flow *f*(*q,t*) for which *dV*/*dt* = 0 (Hicks 1946). In equation (2) Hicksian income is seen to be *rV*.

These arguments support the use of method 2.

Result 1. *Income can be considered to be equal to the current-value Hamiltonian, and depreciation equal to the value of depletion (resource use evaluated at the shadow value), only in a time-autonomous problem. If the problem is not time-autonomous, economic income and depreciation should take into account any capital gains,*  $∂V/∂t$.

Interpretation. *A general method of accounting for income that is consistent with Hotelling and Samuelson's approach to depreciation, records as net product the augmented Hamiltonian,*

$H+λ\dot{S}=H+\frac{∂V}{∂t}=rV$ .

The augmented Hamiltonian is as valid mathematically as the more commonly used Hamiltonian and has some advantages in economic interpretation (cf. Seierstad and Sydsaeter 1977: 373, eq. (24); Silberberg and Suen 2001: ch. 20). The advantages include a direct indication of stock (wealth) equilibrium, in addition to flow equilibrium, through equating to zero the derivative of $H+λ\dot{S}$ with respect to the stock (the adjoint condition): Optimal decisions make the stock at any time exactly what is desired in equilibrium.

Particular results can be derived from the properties of particular RAMs. A frequently studied RAM is the one by which *V*(*S*(*t*),*t*) is maximized. In this case, by the maximum principle *H* is maximized with respect to *q* at each instant of time. If the program is time autonomous, then the capital gain is nil (*∂V/∂t* = 0) and, according to equation (6), the Hamiltonian is equal to the income from the resource, *rV*, and *λq* is the depreciation.

# Four Canonical Optimization Problems

*Hotelling’s Original Problem*

In the simple Hotelling (1931) model with a competitive resource sector and no extraction cost the price at time *t*, *p*(*t*), is exogenous to a given firm’s equilibrium.[[1]](#footnote-1) A firm that plans to extract its stock on the interval  maximizes . The current-value Hamiltonian is

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In the optimal program, *H* = 0 and

.

Furthermore, *V*(*S*(*t*),*t*) = *p*(*t*)*S*(*t*) (Miller and Upton 1985).

As Cairns (2018) argues, . The problem is not autonomous: price (and possibly the interest rate) is exogenous and varies.[[2]](#footnote-2) In its optimization the firm takes the full path *p*(*t*) as given. In the expression for equilibrium value, *p*(*t*)*S*(*t*), price is a function of time. In this case time (or price) is a second state variable. Income is

 ,

and in this case is equal to the capital gain. Depreciation is

.

Depreciation is less than the Hotelling rent, . It is depicted in the plane (*p,S*) in Figure 1. The sign of depreciation depends on the relative sizes of the slivers *pdS* and *Sdp*. For large values of *S*, there may be appreciation: interest on the stock value is greater than the value of the depletion of the stock. The change in *λ* and in *p* is a change in a relative price that directs decisions. Changes in a relative price should be accounted.

In this problem, it is optimal that consumption *not* be equal to Hicksian income no. 1: .[[3]](#footnote-3)

*Maximization of Discounted Utility*

Now let utility be represented by the stationary function, *u*(*q*), and welfare, *W*, by utility discounted at a constant rate *ρ*:

. (7)

Writing down the integral on the RHS of equation (7) implies that utility and welfare are assumed to be real numbers (expressed, say, in the fictive units, *utils per unit time* and *utils*). The assumption invites accounting in terms of utility and welfare, rather than linearizing the Hamiltonian. (The linearized Hamiltonian is still measured in utils per unit time.) Nor is it necessary to introduce a (marginal) utility of money or to account in shadow values.

The value function *W*, a function of the stock *S* alone, is time-autonomous.[[4]](#footnote-4) (No variable is an independent function of time in the optimization problem.) The Hamiltonian is. It is well known that on the optimal path, , so that marginal utility rises at the rate of interest. Depreciation is .

Also, by direct differentiation of the value integral, . Therefore, .

For this optimal, autonomous problem income is equal to the Hamiltonian: , which is equal to the consumers’ surplus, . Depreciation is equal to the Hotelling rent, . The equilibrium at a time *t* is represented in Figure 2.

This problem is often considered to be equivalent to the problem of the competitive industry above. Indeed, the *outcomes* are the same. However, because the consumers’ surplus is incorporated into the present problem and the numeraire is different, the accounting is different: the choice of numeraire is not neutral to the accounting. Income in the competitive industry, *rpS*, is current interest on the market value of the stock. Interest in the welfare optimum is *ρW*, interest (in the form of the rate of time preference) applied to welfare as a stock, and equals the current consumers’ surplus.

The contribution of a competitive equilibrium to welfare is maximized. However, price is exogenous to the firms. Accounting for the firm is as in the Hotelling problem. Formally, for identical firms, the sum of the firms’ accounts (the sectoral account) is not the same as in the welfare accounting, even though the equilibrium paths are the same.

Result 2. *The form of value and the numeraire are not neutral to accounting.*

*Gray’s Problem*

The earliest modern examination of the equilibrium of an exhaustible resource was Lewis C. Gray’s (1914) numerical analysis of the decisions of a competitive mining firm facing stationary, u-shaped average cost and a price *p* that remained constant throughout the life of the mine. Features of Gray’s equilibrium were unveiled by Scott’s (1967) diagrammatic analysis.

(The constant price may appear to be anomalous. In fact, there are two ways to model a competitive sectorial equilibrium as a limit with u-shaped average cost (Cairns 2008). One allows the size of the firm to approach zero as the number of firms increases. This limit eliminates the non-convexity of the problem but allows the market price to increase. The other allows the size of the market to increase such that the market demand becomes horizontal in the limit. This model preserves the non-convexity but at the cost of a horizontal market demand in the limit, and thereby of the constant price. Gray’s problem can be considered to rely on the achievement of the second limit.)

The firm seeks to maximize, for a constant interest rate *r*. The problem is autonomous; therefore, value is a function of the stock alone, *V*(*S*). The level of output decreases from time 0 until, at time *T,* it reaches the level at the minimum point of the average cost curve, when the resource is exhausted. For , the Hamiltonian is

.

For an optimum,

, which rises at rate *r*.

In this autonomous problem, depreciation is the scarcity rent:

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Hicksian income no. 1 is what can be consumed if “capital is maintained intact”, i.e., if . Here, it is possible to maintain capital only if *q* = 0, so that consumption is nil. Hicksian income no. 1 can be defined, but is seen to have no directive significance in this model.

Rather, income is

.

Income and depreciation are depicted in Figure 3.

The income is sometimes called a Ricardian rent to the resource. It is not really a Ricardian rent, however: the resource is homogeneous, not differentiated. Instead, the difference  is attributable to decreasing returns to scale in the technology of extraction, as represented by the cost curve *C*(*q*). Access to (the right to use) the technology is a non-marketed, non-priced asset “held” by the firm. Total value, *V*(*S*(*t*)), is equal to the value of a composite of two assets, the resource and access to the technology. Both assets are essential to production of the output, *q*(*t*), at all times *t* in (0,*T*). Depreciation is the decline in value of the composite, or of what may be called the *mining operation*. Use of the term “Hotelling” rent must be made gingerly then: the scarcity rent (*λ*) is not attributable solely to the resource, even though at the margin it is per unit. The fact that *λ* rises at *r* is called “Gray’s rule” by Cairns (1994) and, especially in light of the exogeneity of price, it (*λ*) might fruitfully be called Gray’s rent.

The problem can be made non-autonomous by postulating that the price, the cost function or the interest rate is a function of time, given exogenously to the firm. Suppose, then, that the firm wishes to maximize

. (8)

Depreciation (of the composite of the resource and access to the technology) is given by ,

which in general is not equal to the aggregate of Gray’s rent,. Income is cash flow net of depreciation:

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Income includes the capital gain, . All sources of drift due to the passage of time – in price, the cost function and the interest rate – contribute to.

Capital is “maintained intact”, so that *V*(*S,t*) does not change, if $\frac{dV}{dt}=\frac{∂V}{∂t}-λq=0$ . Consuming the Hicksian income would take the program off the optimal path, however. Again the concept is not operational for the optimal RAM that has been postulated.

At least in this model, the finding contrasts with one by Wei (2012), who argues that the effects of changes in interest rates should not be used to define Hicksian income. Wei is interested in the definition of income on a constant-consumption path, or in Hicksian income no. 3 (*maximin* income), rather than Hicksian income no. 1. Consuming maximin income maintains value intact when value is defined to be maximin value, .

*Stock Effect*

Now let  and suppose that cost is written *C*(*q*,*Q*), where and . Also let . In an autonomous problem, .

Therefore, .

Depreciation is given by  and income by

.

Income is depicted in Figure 4 as the sum of consumers’ and producers’ surpluses.

(If income is non-autonomous, the values of income and depreciation must be adjusted for the capital gain or loss as above: depreciation is , etc.)

This model represents the depletion, molecule by molecule, of a homogeneous stock of resource. Cairns and Davis (2015) criticize it as not representing an endogenous order of exploitation of distinct qualities “rather than the immanent property of the path assumed by writing down what may be called the *fictive* cost function *C*(*q,Q*)”. In its place, Herfindahl’s (1967) solution is presented as an options equilibrium in which production from each quality is timed to maximize its own scarcity rent,  (which rises at *r* because quality is assumed to be constant within a reserve). These rents are distinguished from the aggregate, Hotelling rent, *λ*. (The Hotelling rent *λ* of the usual stock effect is the envelope of individual scarcity rents  in a limiting process by which the reserves become ever smaller and more numerous.) The price path is taken by the firms as given. In the limit there is no accounting for the producing firm on its infinitesimal life *dt*. The optimal response of firms whose reserves have lower quality than the producing firm is “to wait” before “striking” because the realizable values of their reserves at the equilibrium prices is rising faster than *r*. The rent  of reserve *i* rises at *r* because it includes an option value. Because it is endogenous to the options equilibrium, this type of capital gain should also be credited to reserve *i* as income and product at times before the strike time.

A seemingly sensible and well established model may yield misleading economic-accounting interpretations. More specifically, a higher degree of aggregation may demand greater and more frequent scrutiny.

Result 3. *In sectorial accounting, and a fortiori in national accounting, the representation of equilibrium may have critical implications for accounting.*

# Conclusion

Although it is abstract, the Hotelling model points out some practical considerations for accounting. These problems have been brought out with reference to four optimization problems that have been prominent in the economics of non-renewable resources. Non-autonomy introduces capital gains and losses that must not be neglected because they influence decisions. The choice of the numeraire, the form of the objective and the modeling method have qualitative effects on the accounting because they affect the sources of non-autonomy.

The study of optimal RAMs can be relevant to the study of accounting. Accounting is, however, most fruitfully viewed as providing information in economies in which observed prices are not shadow prices and in which the RAM is not optimal. In real economies, the form of the objective may not be obvious, if one can even hold that there is an intelligible objective. Moreover, it is likely that real economies are non-autonomous, if only because of unaccounted (unaccountable) variables. For these reasons, the simple findings here may have practical significance.

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1. Thanks to Julien Daubanes for this insight. [↑](#footnote-ref-1)
2. Thanks to Julien Daubanes for this point. [↑](#footnote-ref-2)
3. There is an exception where *pdS* = - *Sdp* (where there is unit elasticity of the price with respect to the stock). At this point, value *V = pS*  is maximized and hence *dV/dt = Sdp/dt + pdS/dt =* 0. [↑](#footnote-ref-3)
4. A link to the preceding sectoral equilibrium can be maintained by viewing *u*(*q*) as an increment to total utility and *W*(*S*) as an increment to total welfare. Autonomy may break down. [↑](#footnote-ref-4)