Who pays for growth?

Distributional considerations in a Ramsey model

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Abstract

I study a simple two-sector Ramsey model of capital accumulation or renewable resources exploitation. In the Ramsey spirit, no discount factor is put on future welfares. Growth implies saving, which I interpret as a sacrifice of the current generation at the benefit of future generations. But if two dynasties have access to unequally productive assets, such a sacrifice is not evenly shared among individuals in a given generation. With constant elasticity of substitution specification, I show that the relative sacrifice depends upon two fundamentals elements: the marginal productivity gap and the inequality aversion. In particular, individual consumptions evolve at the same rate only in a case of equal marginal productivity or with an infinite inequality aversion. This result is robust to an unequal inter and intra-dynastic treatment. Finally, links with green accounting are made.

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"...we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination."

Ramsey (1928, p. 543)

1 Introduction

Growth rather than decline is surely an unanimous criterion of economic development. At least as long as this process is sustainable. But for making a production - then implicitly an asset - to grow, we shall not consume our entire income. Or, for a natural resource, we shall not harvest the entire renewal. In a word, we need savings. In an intergenerational perspective, as envisaged by Ramsey (1928), this can be viewed as a sacrifice of current generation to the benefit of future ones. But how this sacrifice should be shared among individuals living in the same generation? Growth theory already answered this question for an aggregate capital and a representative agent through the optimal saving timing. But the question of sharing sacrifice calls for an heterogeneity.

The one-sector Solow-Swan (1956) neoclassical growth model was extended to the two-sector case by Uzawa (1961)[1], but as we know, the saving rate is exogenous. The Ramsey model of optimal saving was extended to the multiple goods-sectors case by Samuelson and Solow (1956). But their approach was not very much followed in the literature. A notable exception that does not collapse to the single sector case was provided by Pitchford (1977). But the two sectors, each in one region of the economy, produce one single good. Rather, two-sector economy was mainly analyzed through the discounted version of the Ramsey model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) (RCK hereafter). This was the case to deal with, for instance, physical capital and exhaustible resources (Dasgupta and Heal, 1974) or physical and human capital (Lucas, 1988).[2] On the other hand, heterogeneity of agents was analyzed both in the Solow-Swan model (Stiglitz, 1969) and in the RCK model, through different individual discount rates (Ramsey, 1928; Becker, 1980; Becker and Foias, 1987) or through different initial endowments of wealth (Chatterjee, 1994; Caselli and Ventura, 2000)[3]. Heterogeneity of both sectors and agents in the RCK model was proposed by Becker and Tsyganov (2002), where there are one capital good and one production good that can be aggregated. More broadly, both intra and intergenerational considerations of sharing resources can be found in the growth-inequality literature.

Even if trends of long-term economic development and distribution of wealth was analyzed

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[1]See also Uzawa (1963) and the references mentioned.

[2]In a sense, a two-sector RCK model was earlier proposed by Srinivasan (1964) and Uzawa (1964), but in an non-utilitarian approach.

[3]See also Garcìa-Peñalosa and Turnovsky (2012) and the references in there.

by the classical economists in the XIXth century, it was first analyzed extensively with use of data by Kuznets (Kuznets and Jenks (1953); Kuznets (1955)). According to him, inequalities rise, then decrease, in the process of development. This topic was further analyzed by Atkinson and Harrison (1978), and current works refute the Kuznets hypothesis (e.g. Piketty, 2015). For a review of the growth-inequality relationship see Aghion et al. (1999)[4].

Though linked, my question here is different in the sense that I am interested in linking ethical judgments with outcomes of the distribution, both between individuals living at the same time and between different generation living, by definition, at different dates. This is the so-called intra and intergenerational equity relationship (see Isaac and Piacquadio (2015), Kverndokk et al. (2014) and the references in there). Equity requires here equal weights[5], both between two individuals in intra, and between two generations in inter.

On the intergenerational dimension, the intertemporal social welfare criterion has to respect finite anonymity: a finite number of permutation in the time of appearance of two generations shall not change the result. This is supported by a long tradition in economics against discounting future welfares. Famous examples are those of Sidgwick (1907, p. 414), Ramsey (1928, p. 543), Pigou (1920, pp. 24-25), Harrod (1948, p. 40), Solow (1993, p. 165). Unfortunately, as shown by Koopmans (1960), if a (social) criterion of infinite utility stream respect some *a priori* desirable axioms, it has to exhibit "impatience", i.e. discounting. Following his approach, Diamond (1965) stated a classic impossibility result: such a criterion cannot both be efficient (strong Pareto) and treat all generations equally. For a critical survey of related results, see Asheim (2010). The Ramsey (1928) approach respect both "efficiency" and "equity" but is incomplete.

The intragenerational equity can be understood in several ways, see for example Fleurbaey and Maniquet (2011). But, far from reaching exhaustiveness, a *symmetric* social welfare function *a la* Bergson (1938)-Samuelson (1947) can represent different equity views of a society.

Here I study sacrifice - understood as the difference between the sustainable individual maximal consumption and the actual consumption - made by different individuals when one wishes to reach a higher social well-being. I model two individuals, each one has access to a different stock, as first modeled by Samuelson and Solow (1956). They can represent an asset to which only some individuals have access or two renewable resources at different locations that one can harvest without cost. It may also represent two countries ("North" and "South") that can exchange without cost. The intratemporal social preferences are represented by a social welfare function with a constant elasticity of substitution (CES). As argued by Atkinson (1970), the parameter of elasticity may be interpreted as a parameter of inequality aversion. Indeed, such a

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[4]A more recent review can be found, in French, in Garcìa-Peñalosa (2017).

[5]There is not a unique accepted definition of the word "equity". Our requirement here is in line with a dictionary definition: "Equity is the quality of being fair and reasonable in a way that gives equal treatment to everyone." (source: `https://www.collinsdictionary.com/dictionary/english/equity`, visited on May 2017).

function encompasses two famous special cases: pure utilitarianism (elasticity goes to infinity) and the symmetric minimum, sometimes called "rawlsian" (elasticity goes to zero). Symmetrically, the intertemporal social preferences are represented by an intertemporal social welfare function with a constant intertemporal elasticity of substitution (CIES). More precisely, I minimize the difference between the welfare targeted and the actual welfare, in the spirit of Ramsey (1928). Here also, the parameter of intertemporal elasticity may be interpreted as a parameter of intertemporal inequality aversion. As in its instantaneous counterpart, such a function can tend to a nil inequality aversion case (elasticity goes to infinity) or to an infinite inequality aversion case (elasticity goes to zero). Interestingly, this latter case may correspond to the maximin (d'Autume and Schubert, 2008a), which has resonance with assessing sustainability (Cairns and Martinet, 2014; Fleurbaey, 2015).

I show that when society has to make an intergenerational sacrifice, two main elements guide its sharing. The marginal productivity gap and the intragenerational inequality aversion. The former indicates that an individual having access to a relatively more productive asset may afford a relative higher consumption. Paradoxically, the individual having access to the relatively less productive asset has to make a higher sacrifice. This comes from the fact that s/he cannot both have a high consumption and a high growth of his/her stock. The impact of the productivity gap on individual consumption growth rates is weighted by the inequality aversion. The more society is willing to substitute individuals welfare, the more one can take advantage of their difference, then the more is the difference in terms of sacrifice. At the opposite, when society exhibits an infinite intragenerational inequality, individual growth rates are equal, whatever the productivity gap.

The next Section introduces the model: first in isolation, individuals are collectively considered afterward. Section 3 exhibits links with the green accounting literature. I allow for an unequal treatment of individuals in Section 4 . Section 5 concludes.

2 Adressing Equity with the Ramsey Device

2.1 Notations

I study an economy composed of two dynasties with an infinite number of generations, each one consumes c_i of the asset $i = 1, 2$. The production function is assumed to exhibit decreasing return to scale, such that the net production F_i is increasing, strictly concave and reaches a maximum at $\bar{X}_i$, the golden-rule stock. This can represent either the direct renewal of a renewable resources[6] or the production of a good, net of the depreciation. The dynamics of

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[6]I recognize that the dynamics of some resources cannot be represented by such a function. This is a stylized feature.

the stock is then given by (the time index is dropped hereafter since no confusion arises)

$$\frac{dX_i(t)}{dt} \equiv \dot{X}_i(t) = F_i(X_i(t)) - c_i(t) , \quad i = 1, 2 . \tag{1}$$

I consider only the policy-maker program. This is in line with my problematic because I am not interested in the intra-dynastic flow of utility, but rather in how should evolves the aggregate social welfare. For simplicity, utility functions are assumed to be linear: $u_i(c_i) = c_i$. In a perspective of risk, it represents a nil risk-aversion. It also depicts the fact that, from a social planer point of view, individuals are responsible for their utility.

The instantaneous welfare is given by[7]

$$U(c_1, c_2) = \left(\frac{1}{2} \cdot c_1^\eta + \frac{1}{2} \cdot c_2^\eta \right)^{\frac{1}{\eta}} , \quad \eta < 1 , \eta \neq 0 . \tag{2}$$

The social elasticity of substitution is given by $\theta \equiv \frac{1}{1-\eta}$. Which can be interpreted as a parameter of inequality aversion (Atkinson, 1970): from 0 (infinite inequality aversion) to infinity (nil inequality aversion).

The social welfare function is given by[8]

$$W(c_1, c_2) = \frac{U(c_1, c_2)^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} , \quad \sigma > 0 , \sigma \neq 1 . \tag{3}$$

The intertemporal elasticity of substitution is given by σ. Which can be thought as a measure of intertemporal inequality aversion. As in a static framework, W tends to be linear or (is equivalent to) the minimum form when σ tends respectively to positive infinity or zero (see the Appendix).

2.2 Self-sufficiency

The economy is entitled of two initial stocks given by $(X_1(0), X_2(0)) = (X_{10}, X_{20})$. Let us suppose each individual consumes the entire production (or renewal) of the asset s/he has access. If the initial stock is higher that the golden-rule stock $\bar{X}_i$ (maximum sustainable yield), I suppose s/he consumes the corresponding production, and the stock decreases to the golden-rule stock. To sum up:

$$c_i^\# = \begin{cases} F_i(X_{i0}) & \text{if} \quad X_{i0} \leq \bar{X}_i ; \\ F_i(\bar{X}_{i0}) & \text{if} \quad X_{i0} > \bar{X}_i . \end{cases} \tag{4}$$

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[7]The weights $\frac{1}{2}$ (resp. the -1 in its intertemporal counterpart) are there only for having the Cobb-Douglass (the logarithm) as a special case in the limit.

[8]Even if they were not concerned by the same subject, this approach is similar to the one proposed by d'Autume and Schubert (2008a) and d'Autume et al. (2010).

This policy is sustainable, and four cases may be distinguished.

	$X_{10} \leq \bar{X}_1$	$X_{10} > \bar{X}_1$
$X_{20} \leq \bar{X}_2$	A	B
$X_{20} > \bar{X}_2$	C	D

Case A is the most intuitive, in which each individual consumes its production. The social welfare is $W^{\#} = W(F_1(X_{10}), F_2(X_{20}))$. Case D depicts an abundant world, where the individual consumptions are at their highest sustainable level. Consumptions may even, momentarily, be higher than the golden-rule production.[9] Cases B and D represent an asymmetric situation: an agent is limited to its production, while another can consume more than the golden-rule.

Individually, the agents do not have interest in exchange. But from a social point of view, exchange can only done better results. In the case A, if marginal productivities are different, exchanges can take advantage of this difference to save/dissave to give a higher social welfare, even if it stays constant over time.[10] And making them both save, economy grows. In the case D, exchanges would give the same result. In the cases B or C, exchanges would unambiguously improve the social welfare compared to the autarky situation. The agent with an abundant stock could over-consume while the agent with a scarce stock under-consumes to let the latter stock grow. In such a case, the welfare rises (as in case A). But the former stock may be so abundant compared to the latter, that the welfare could even be momentarily higher than its sustainable level $\bar{W} = W(F_1(\bar{X}_1), F_2(\bar{X}_2))$ (as in case D).

When exchanges are possible and except in a case of abundance (absolute or relative), growth imply a sacrifice. Which can be measured as the difference between the maximum sustainable consumption (c^*) and the actual consumption (c).

2.3 Sacrifice for growth

Let us assume the social planner wishes to attain $W^f = W\left(X_1^f, X_2^f\right)$. Naturally, the final welfare is at least as high as the maximum sustainable one (maximin). This implies a growth of welfare (except if initial stocks are higher than the as good-as-golden locus).

Let the social planner minimize the global sacrifice, while targeting a higher final welfare than the maximum sustainable one reachable from initial stocks. I minimize the distance between the maximum sustainable welfare and the current one. This is equivalent to minimize the distance between the targeted welfare and the current one since, here, current welfare cannot exceed the sustainable one.[11] This is done by the Ramsey's objective. The intertemporal

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[9]In the literature on growth, this is know as "dynamic inefficiency".

[10]This the case of a maximin policy, see Cairns et al. (2016).

[11]Formally: current welfare $\leq$ maximum sustainable welfare $\leq$ targeted welfare.

welfare is given by[12]

$$V(X_1, X_2) = \max_{c_1, c_2} \int_0^\infty \left(W(c_1, c_2) - W^f \right) dt \; ; \tag{5}$$

$$\text{subject to} \quad \dot{X}_1 = F_1(X_1) - c_1 \; ; \tag{6}$$

$$\dot{X}_2 = F_2(X_2) - c_2 \; ; \tag{7}$$

$$(X_{10}, X_{20}) \text{ given} \; . \tag{8}$$

I assume that the utility evolves fast enough such that the integral is finite (discussion on convergence can be found in Chiang (1992, pp. 99-101)). The Hamiltonian is

$$\mathcal{H}(X_i, c_i, \psi_i) = W(c_1, c_2) - W^f + \psi_1(F_1(X_1) - c_1) + \psi_2(F_2(X_2) - c_2), \quad i = 1, 2 \; . \tag{9}$$

The necessary conditions contain:

$$\frac{\partial \mathcal{H}(X_i, c_i, \psi_i)}{\partial c_i} = 0 \quad \Leftrightarrow \quad \psi_i = U^{-\frac{1}{\sigma}} \cdot U_{c_i}, \quad i = 1, 2 \; ; \tag{10}$$

$$-\frac{\partial \mathcal{H}(X_i, c_i, \psi_i)}{\partial X_i} = \dot{\psi}_i \quad \Leftrightarrow \quad -\frac{\dot{\psi}_i}{\psi_i} = F_i'(X_i), \quad i = 1, 2 \; ; \tag{11}$$

$$\lim_{t \to \infty} \mathcal{H}(X_i, c_i, \psi_i) = 0, \quad i = 1, 2 \; . \tag{12}$$

As usual, the shadow-price of a stock equal its marginal value in terms of welfare (eq. (10)). But here, as there is no time preference, it will always pay to sacrifice ($\dot{c}_i > 0 \Leftrightarrow \dot{\psi}_i < 0$) as long as the marginal return on capital is positive (eq. (11)). From the equations (10) and (11), we have

$$\frac{1}{\sigma} \frac{\dot{U}}{U} - \frac{\dot{U}_{c_i}}{U_{c_i}} = F_i'(X_i) \quad \Leftrightarrow \quad \frac{\dot{U}}{U} = \sigma \left(F_i'(X_i) + \frac{\dot{U}_{c_i}}{U_{c_i}} \right), \quad i = 1, 2 \; . \tag{13}$$

The evolution of the instantaneous welfare depends positively on the intertemporal elasticity of substitution, since it represents a higher willingness to substitute current and future welfares. Society save more today in order to have a higher growth in the future. At the limit, it becomes constant when the elasticity of substitution approaches zero. With a CES instantaneous social welfare function, the previous equation can be expressed as (see Appendix A.3)

$$\frac{\dot{U}}{U} = \sigma \left(F_i'(X_i) + \frac{1}{\theta} \left(Z - \frac{\dot{c}_i}{c_i} \right) \right), \quad Z \equiv \frac{1}{2} \left(\left(\frac{c_i}{U} \right)^\eta \frac{\dot{c}_i}{c_i} + \left(\frac{c_j}{U} \right)^\eta \frac{\dot{c}_j}{c_j} \right), \quad i \neq j \; . \tag{14}$$

Since this holds for each stock, equalizing them and rearranging terms gives

$$\theta \left(F_i' - F_j' \right) = \frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j} \; . \tag{15}$$

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[12]As a by-product of our analysis, the 'abundance' case - when initial stocks are higher than the golden-rule - is also handled: the objective maximize the current welfare, net of its lower limit.

Relatives growth of individual consumptions depend on the difference of marginal productivities, weighted by the intragenerational inequality aversion. For example, the higher the marginal productivity of the i's stock, the higher his/her consumption growth rate. Let us interpret the global individual sacrifice by the difference between final and current consumption. I can say that sacrifice of i is relatively lower than the sacrifice of j. The latter, while having access to a less productive stock has to make a relative higher sacrifice. The reason is that j's stock needs relative high investments to rise. What cannot be done maintaining a high consumption. For the opposite reason, i can afford a relative high consumption during the transition. At the limit, when the elasticity tends to zero, consumptions evolves at the same rate. And when it tends to infinity, social welfare is nearly only supported by the consumption of the individual who has access to the more productive stock, while the other let his/her stock grow (consumptions approach a corner solution).

Using the transversality condition, a steady state is characterized by (the stability is proved in the Appendix A.4)

$$c_1^f = F_1\left(X_1^f\right) , \quad c_2^f = F_2\left(X_2^f\right) , \quad F_1'\left(X_1^f\right) = F_2'\left(X_2^f\right) , \quad W(c_1,c_2) = W^f . \quad (16)$$

Approaching the steady state (possibly asymptotically), stock and consumption path have the same tangent (by L'Hospital's rule):

$$\lim_{t\to\infty} \frac{\mathrm{d}X_2}{\mathrm{d}X_1} = \lim_{t\to\infty} \frac{F_2'(X_1)\dot{X}_2 - \dot{c}_2}{F_1'(X_1)\dot{X}_1 - \dot{c}_1} = \frac{\mathrm{d}c_2}{\mathrm{d}c_1} . \quad (17)$$

Integrating both sides of the equation (15) and using the final consumptions, it comes

$$\frac{c_i}{c_j} = \frac{c_i^f}{c_j^f} \mathrm{e}^{\theta \int_0^\infty \left(F_i' - F_j'\right)\mathrm{d}t} . \quad (18)$$

It is worth noticing that the intertemporal inequality aversion plays no role on the evolution of the relative consumptions. This comes from the fact that the intertemporal elasticity has the same impact on each consumer living at the same time. More precisely, both inter and intra elasticities impact consumptions: $c_i = f(\theta,\sigma)$, but only the intra impacts the ratio: $\frac{c_i}{c_j} = f(\theta)$.

I turn to a graphical illustration to better understand the features of the system. I assume, for simplicity, that the the targeted welfare is the one reached as the golden rule: $W^f = \bar{W}$. Following Cairns et al. (2016), the Fig. 1 is a four-quadrant plot that represents the different situations. The North-East quadrant plots the evolution of the consumption and the stock of the first agent. Symmetrically, the South-West plots these elements for the second agent. In such a way, the North-West quadrant plots the consumption paths. Finally, the South-Est quadrant plots the stock paths.
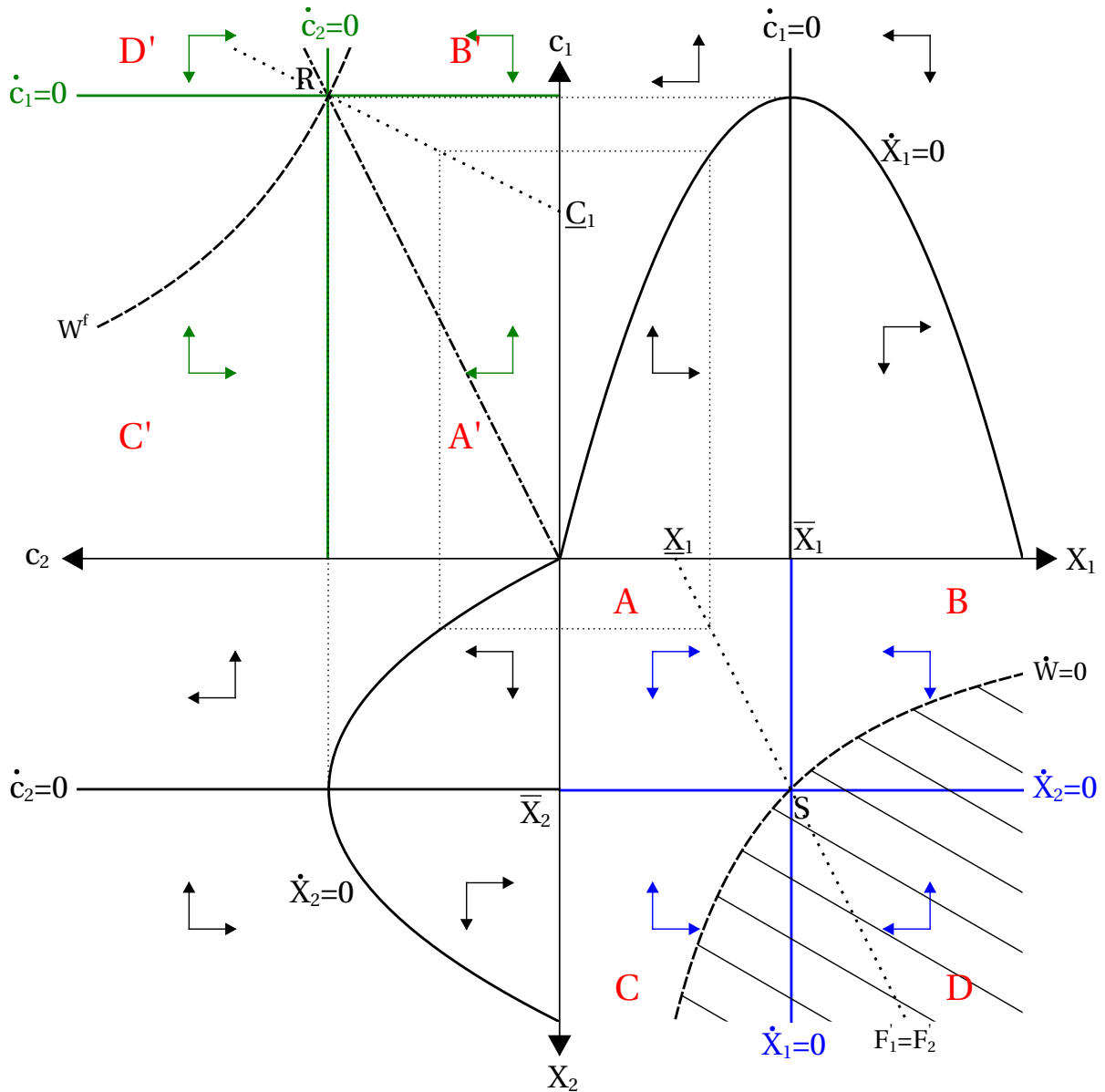
Figure 1: Global evolutions of consumptions and stocks

NE and SW quadrants are, individually, similar to a one-sector RCK model with no discount factor.[13] It depicts a saddle point: consumption has to rise (resp. decrease) if the initial stock is lower (higher) than the golden-rule stock. For the sake of the representation, I use the transversality condition (reaching the steady state) to project the $\dot{c}_i = 0$ curves in the NW and the $\dot{X}_i = 0$ curves in the SE. NW and SE quadrants represent only optimal paths, ending up at the steady state (R and S). The paths never cross the $F'_1 = F'_2$ locus[14] ($\underline{X}_1 S$ and $\underline{C}_1 R$), otherwise one would have a steady state that does not respect the transversality condition. I also conjecture that paths never cross the curve passing through S (R) depicting $\dot{W} = 0$ (W^f).[15] This curve represents the

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[13]It is direct from the eq. (10) and (11), noticing that $\psi_i > 0$ is a decreasing function of U_{c_i}.

[14]It is a straight line because I plotted symmetric production functions.

[15]A discussion on this point can be found in Samuelson and Solow (1956, p. 550).

as-good-as-golden locus (Phelps and Riley, 1978). The hatched are depicts stocks from which ones, the social welfare decrease towards its steady state. More generally, no cycle occurs (in consumptions or states) since path are asymptotic to the 'critical loci'.

Let us go back to our four situations of the beginning. The case A (corresponding consumption areas are noted with an apostrophe) need a sacrifice of the two agents. The case D represents an excess of consumptions compared to their final level: the welfare declines to reach its sustainable value. Cases B and D are more interesting. One takes advantage on the more abundant stock (opportunity cost) to let the other one grow. In C, for example, the sacrifice of the agent 1 is made possible by the overconsumption of the agent 2. If the first stock is very abundant (hatched area of the case C), the welfare is higher than its sustainable level, and then decreases, as in the case D. The overconsumption by an agent (dynasty) in case B and C could be used to compensate the agent making a sacrifice.

3   Green Accounting

In autonomous problems, the optimal Hamiltonian is constant (see for example Chiang (1992, p. 190)). The transversality condition (12) implies therefore $\mathscr{H}^* = 0$. The equation (9) can then be rewritten as

$$\underbrace{\psi_1 \dot{X}_1 + \psi_2 \dot{X}_1}_{\text{Net investment}} = \underbrace{W^f - W}_{\text{Welfare gap}} \ . \tag{19}$$

Along a path, as long as $W < W^f$, net investment (NI) is positive and is linked to the "welfare distance". The farther the economy is from the target, the more we have to invest. And it tends towards zero as W approaches W^f. When the current welfare equal the targeted welfare (possibly asymptotically), we get the Hartwick's rule (Hartwick, 1977; Dixit et al., 1980; Withagen and Asheim, 1998). Actually, this feature was found by Ramsey (1928) himself and was restated with exhaustible resource and capital accumulation by d'Autume and Schubert (2008b). Here, positive, nil and negative investments are possible according to the initial states.

Let us consider the maximin policy of the same problem, from the same initial states and on the same planning horizon (this can be found in Cairns et al. (2016)). Along a path, the maximin value m evolves.[16] In particular $\dot{m}(X_1, X_2) = \mu_1 \dot{X}_1 + \mu_2 \dot{X}_2$, where μ_i are maximin shadow prices (Cairns and Martinet, 2014, Lemma 1). We state the following proposition linking NI and the maximin value improvement.

Proposition 1 (Ramsey Net Investment)**.** *An optimal path of the problem* (22) *subject to the*

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[16]Paradoxically, taking "Bliss" (W^f here) as the maximin value, the Ramsey rule generalizes the Hartwick's one.

constraints (6)-(8) is such that

$$\psi_1 \dot{X}_1 + \psi_2 \dot{X}_2 \geq 0 \quad \Leftrightarrow \quad \dot{m}(X_1, X_2) \geq 0 \,. \tag{20}$$

Without surprise, in a scarce world, sacrifice allows to increase the sustainable welfare.

4 Inter and intra inequity

As Ramsey (1928), I now consider discounting. I also attach different weights on individuals. I interpret these weights (inter and intra) as a measure of "inequity".

The instantaneous welfare becomes

$$\tilde{U}(c_1, c_2) = \left(\alpha \cdot c_1^\eta + (1 - \alpha) \cdot c_2^\eta \right)^{\frac{1}{\eta}}, \ 0 < \alpha < 1 \,. \tag{21}$$

The social welfare function is unchanged. For a positive discount rate δ, the intertemporal welfare is now given by[17]

$$\tilde{V}(X_1, X_2) = \max_{c_1, c_2} \int_0^\infty e^{-\delta t} \left(W(c_1, c_2) - W^f \right) dt \,; \tag{22}$$

$$\text{subject to the same constraints .} \tag{23}$$

The eq. (13) becomes (the other conditions are unchanged)

$$\frac{\dot{U}}{U} = \sigma \left(F_i'(X_i) + \frac{\dot{U}_{c_i}}{U_{c_i}} - \delta \right), \quad i = 1, 2 \,, \tag{24}$$

which is the same equation of the evolution of the utility of d'Autume and Schubert (2008a). Impatience reduces the utility growth rate, future welfares are indeed less valued. But, as in its intra counterpart, when intertemporal elasticity of substitution trends to zero, weights disappear. They become "immaterial" (d'Autume and Schubert, 2008b, p. 840).

Even if the eq. (14) changes to exhibits weights, the eq. (15) remains the same. Interestingly, the result on evolution of individual consumptions is robust both to inter and intra "inequity".

5 Concluding remarks

Even simple, this framework enables us to have straightforward insights. When dynasties of individuals have access to unequally productive assets - which can be physical capital or renewable resources - sacrifice to attain a better sustainable level of welfare is not evenly shared.

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[17]Koopmans (1965) used a similar formulation to connect discounted and undiscounted versions.

Those who can take advantage of a relative less productive stock have to make a relatively higher sacrifice in order to compensate the low growth potential of their stock. This holds even if unequal weights are put on individuals and between generations (discounting). However, if society has an infinite intragenerational inequality aversion, relative sacrifice is evenly shared. And if the social planner has an infinite intergenerational inequality aversion, the social welfare remains constant, avoiding intertemporal sacrifice.

These results are in line with the literature on growth and inequality. For example, Aghion et al. (1999) stated that: "*redistribution can foster growth. However, the growth process is unlikely to leave inequality unchanged*". A natural question that arises is about the trade-off between time of sacrifice and level of welfare targeted.

A   Appendix: Proofs and calculus

A.1   Proof of the Proposition

Proof of the Proposition 1. By construction, $W \leq W^f$. From the equation (19), we have

$$\psi_1 \dot{X}_1 + \psi_2 \dot{X}_1 \geq 0 \quad \Leftrightarrow \quad \dot{X}_1 + \frac{\psi_2}{\psi_1} \dot{X}_1 \geq 0 \,. \tag{25}$$

From Cairns and Martinet (2014, Lemma 1): $\dot{m}(X_1, X_2) \geq 0 \Leftrightarrow \dot{X}_1 + \frac{\mu_2}{\mu_1} \dot{X}_2 \geq 0$.[18]
From Cairns et al. (2016) (substituting U by W): $\frac{\mu_2}{\mu_1} = \frac{W_2}{W_1} = \frac{U_2}{U_1}$.
As $\frac{\psi_2}{\psi_1} = \frac{U_2}{U_1}$ from the equation (10), we have

$$\psi_1 \dot{X}_1 + \psi_2 \dot{X}_1 \geq 0 \quad \Leftrightarrow \quad \dot{m}(X_1, X_2) \geq 0 \,. \tag{26}$$

$\square$

A.2   Special Cases of the CIES Welfare Function

This proof is mainly concerned with the restatement of the result of d'Autume and Schubert (2008a) in a zero-discounting framework.

Proof. Let us consider a continuously differentiable function with a constant intertemporal elasticity of substitution (CIES) $\sigma \equiv \frac{1}{1-\rho}$: $f(x) = \frac{x^\rho - 1}{\rho}$. With $\rho < 1$, $\rho \neq 0$. Notice that σ is the inverse of the relative risk aversion (which is given by the opposite of the elasticity of marginal utility[19]).[20]

- Case 1: the elasticity tends to positive infinity. Trivially, if $\theta \to \infty$ ($\rho \to 1$), the CIES function tends to a linear function[21]

$$\lim_{\rho \to 1} f(x) = x - 1 \,. \tag{27}$$

- Case 2: the elasticity tends to zero ($\rho \to -\infty$). Let us assume

$$\int_0^\infty \left(f(x) - \bar{f} \right) \mathrm{d}t < \infty \,. \tag{28}$$

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[18]See Cairns and Long (2006) for conditions ensuring $\mu_i > 0$.

[19]$\mathrm{IES} := \frac{\mathrm{dln}(x)/\mathrm{d}t}{-\mathrm{dln}(f'(x))/\mathrm{d}t} = \frac{\frac{\dot{x}}{x}}{-\frac{f''(x)\dot{x}}{f'(x)}} = -\frac{f'(x)}{x f''(x)}$.

[20]Interested readers can see Hall (1988).

[21]The negativity of utility for $0 < x < 1$ does not pose any difficulty in a marginalist approach. If needed, the function $f(x) = \frac{(1+x)^\rho - 1}{\rho} \geq 0$ can be considered.

The-utilitarian form (with the discount factor $\delta > 0$) is equivalent to maximize the zero-discount one when the discount factor tends to zero:

$$\lim_{\delta \to 0} \int_0^\infty e^{-\delta t} \left(f(x) - \bar{f} \right) dt = \int_0^\infty \left(f(x) - \bar{f} \right) dt . \tag{29}$$

As shown by d'Autume and Schubert (2008a, pp. 272-273), maximizing the discount-utilitarian form is equivalent to maximize the minimum form when the elasticity of substitution tends to zero:[22]

$$\operatorname{argmax} \lim_{\rho \to -\infty} \int_0^\infty e^{-\delta t} f(x) dt = \operatorname{argmax} \min_t f(x) . \tag{30}$$

Therefore, under the assumption (28):

$$\operatorname{argmax} \lim_{\rho \to -\infty} \int_0^\infty \left(f(x) - \bar{f} \right) dt = \operatorname{argmax} \min_t f(x) .^{23} \tag{31}$$

- Case 3: for exhaustiveness, let the elasticity tend to one. Let $g_1(\rho)$ and $g_2(\rho)$ be equivalent, respectively, to the numerator and to the denominator. As $\lim_{\rho \to 0} g_1(\rho) = \lim_{\rho \to 0} g_2(\rho) = 0$, by the Hospital's rule:

$$\lim_{\rho \to 0} \frac{g_1(\rho)}{g_2(\rho)} = \lim_{\rho \to 0} \frac{g_1'(\rho)}{g_2'(\rho)} , \tag{32}$$

and as

$$\frac{g_1'(\rho)}{g_2'(\rho)} = \frac{\ln(x) x^\rho}{1} , \tag{33}$$

one gets finally

$$\lim_{\rho \to 0} f(x) = \ln(x) , \quad x > 0 . \tag{34}$$

□

A.3 CES assumption

Let us explicit the logarithmic time derivative of the marginal utility:

$$\frac{\dot{U}_{c_i}}{U_{c_i}} = \frac{U_{c_i c_i}}{U_{c_i}} \dot{c}_i + \frac{U_{c_i c_j}}{U_{c_i}} \dot{c}_j ; \tag{35}$$

$$\frac{\dot{U}_{c_j}}{U_{c_j}} = \frac{U_{c_j c_i}}{U_{c_j}} \dot{c}_i + \frac{U_{c_j c_j}}{U_{c_j}} \dot{c}_j . \tag{36}$$

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[22]With our notations, their function is $h(x) = \frac{x^\rho}{\rho}$, but clearly $\max f(x) \equiv \max h(x)$.

[23]This justifies the sentence: "*zero discounting criteria [...] includes maximin as a particular case or, more precisely, as the limit case of a zero elasticity of substitution between the welfares of different generations.*" (d'Autume et al., 2010, p. 194).

Let us compute each partial derivative, assuming a CES utility function. To state the result in generals terms, we put indefinite weights on the different consumptions (letting $\alpha = \frac{1}{2}$ give us our special case): $U\left(c_i, c_j\right) = \left(\alpha \cdot c_i^\eta + (1-\alpha) \cdot c_j^\eta\right)^{\frac{1}{\eta}}$, $0 < \alpha < 1$.

First-order derivatives are:

$$U_{c_i} = \alpha \cdot U^{1-\eta} \cdot c_i^{\eta-1} \ ;$$
$$U_{c_j} = (1-\alpha)U^{1-\eta} \cdot c_j^{\eta-1} \ .$$

Second-order derivatives are:

$$U_{c_i c_i} = \alpha(1-\eta)U^{1-\eta}c_i^{\eta-2}\left(\frac{\alpha \cdot c_i^\eta}{U^\eta} - 1\right) \ ;$$

$$U_{c_j c_j} = (1-\alpha)(1-\eta)U^{1-\eta}c_j^{\eta-2}\left(\frac{(1-\alpha)c_j^\eta}{U^\eta} - 1\right) \ ;$$

$$U_{c_i c_j} = U_{c_j c_i} = \alpha(1-\alpha)(1-\eta)U^{1-2\eta} \cdot c_i^{\eta-1} \cdot c_j^{\eta-1} \ .$$

Then:

$$\frac{U_{c_i c_i}}{U_{c_i}} = \frac{\alpha(1-\eta)U^{1-\eta}c_i^{\eta-2}\left(\frac{\alpha \cdot c_i^\eta}{U^\eta} - 1\right)}{\alpha \cdot U^{1-\eta} \cdot c_i^{\eta-1}} = \frac{1-\eta}{c_i}\left(\alpha\left(\frac{c_i}{U}\right)^\eta - 1\right) \ ;$$

$$\frac{U_{c_i c_j}}{U_{c_i}} = \frac{\alpha(1-\alpha)(1-\eta)U^{1-2\eta} \cdot c_i^{\eta-1} \cdot c_j^{\eta-1}}{\alpha \cdot U^{1-\eta} \cdot c_i^{\eta-1}} = \frac{1-\eta}{c_j}(1-\alpha)\left(\frac{c_j}{U}\right)^\eta \ ;$$

$$\frac{U_{c_j c_i}}{U_{c_j}} = \frac{\alpha(1-\alpha)(1-\eta)U^{1-2\eta} \cdot c_i^{\eta-1} \cdot c_j^{\eta-1}}{(1-\alpha)U^{1-\eta} \cdot c_j^{\eta-1}} = \frac{1-\eta}{c_j}\alpha\left(\frac{c_i}{U}\right)^\eta \ ;$$

$$\frac{U_{c_j c_j}}{U_{c_j}} = \frac{(1-\alpha)(1-\eta)U^{1-\eta}c_j^{\eta-2}\left(\frac{(1-\alpha)c_j^\eta}{U^\eta} - 1\right)}{(1-\alpha)U^{1-\eta} \cdot c_j^{\eta-1}} = \frac{1-\eta}{c_j}\left((1-\alpha)\left(\frac{c_j}{U}\right)^\eta - 1\right) \ .$$

The equations (35) and (36) become

$$\frac{\dot{U}_{c_i}}{U_{c_i}} = (1-\eta)\left(\left(\alpha\left(\frac{c_i}{U}\right)^\eta - 1\right)\frac{\dot{c}_i}{c_i} + (1-\alpha)\left(\frac{c_j}{U}\right)^\eta \frac{\dot{c}_j}{c_j}\right) \ ; \qquad (37)$$

$$\frac{\dot{U}_{c_j}}{U_{c_j}} = (1-\eta)\left(\alpha\left(\frac{c_i}{U}\right)^\eta \frac{\dot{c}_i}{c_i} + \left((1-\alpha)\left(\frac{c_j}{U}\right)^\eta - 1\right)\frac{\dot{c}_j}{c_j}\right) \ . \qquad (38)$$

With $\alpha = \frac{1}{2}$:

$$\frac{\dot{U}_{c_i}}{U_{c_i}} = (1-\eta)\left[\underbrace{\frac{1}{2}\left(\left(\frac{c_i}{U}\right)^{\eta}\frac{\dot{c}_i}{c_i} + \left(\frac{c_j}{U}\right)^{\eta}\frac{\dot{c}_j}{c_j}\right)}_{\equiv Z} - \frac{\dot{c}_i}{c_i}\right], \quad i \neq j. \tag{39}$$

A.4 Stability of the steady state

Let us sum up the necessary conditions into three main equations to get the following dynamic system, with $Y \equiv \ln\left(\frac{c_2}{c_1}\right)$;

$$\begin{cases} \dot{X}_1 &=& F_1(X_1) - c_1\,; \\ \dot{X}_2 &=& F_2(X_2) - c_2\,; \\ \dot{Y} &=& \theta\left(F_2'(X_2) - F_1'(X_1)\right). \end{cases} \tag{40}$$

Steady states are characterized by (using the transversality condition)

$$\begin{cases} c_1^f &=& F_1\left(X_1^f\right)\,; \\ c_2^f &=& F_2\left(X_2^f\right)\,; \\ F_1'\left(X_1^f\right) &=& F_2'\left(X_2^f\right). \end{cases} \tag{41}$$

Consider the Jacobian matrix of the linearized system, evaluated at the steady states[24]

$$J^{\star} = \begin{pmatrix} F_1' & 0 & \alpha_1 \\ 0 & F_1' & -\alpha_2 \\ -\theta F_1'' & \theta F_2'' & 0 \end{pmatrix}; \tag{42}$$

with $\alpha_1 \equiv -\left.\frac{\partial c_1}{\partial Y}\right|_{c_1^f, c_2^f} = \frac{\left(c_2^f\right)^2}{c_1^f} > 0$ and $\alpha_2 \equiv \left.\frac{\partial c_2}{\partial Y}\right|_{c_1^f, c_2^f} = c_2^f > 0.$

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[24]We use the equality $F_1' = F_2'$ and express the Jacobian with respect to F_1' only.

Let us compute the roots of the characteristic polynomial $\mathscr{P}(\lambda) = \det\left(J^f - \lambda I_3\right)$:

$$\begin{vmatrix} F_1' - \lambda & 0 & \alpha_1 \\ 0 & F_1' - \lambda & -\alpha_2 \\ -\theta F_1'' & \theta F_2'' & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \qquad \left(F_1' - \lambda\right) \begin{vmatrix} F_1' - \lambda & -\alpha_2 \\ \theta F_2'' & -\lambda \end{vmatrix} - \theta F_1'' \begin{vmatrix} 0 & \alpha_1 \\ F_1' - \lambda & -\alpha_2 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \left(F_1' - \lambda\right)\left(-\left(F_1' - \lambda\right)\lambda + \theta F_2'' \alpha_2\right) + \theta F_1'' \left(F_1' - \lambda\right)\alpha_1 = 0$$

$$\Leftrightarrow \qquad \left(F_1' - \lambda\right)\left(-\left(F_1' - \lambda\right)\lambda + \theta \alpha_1 F_1'' + \theta \alpha_2 F_2''\right) = 0. \qquad (43)$$

The first eigenvalue is $\lambda_1 = F_1'$. Let $\Gamma \equiv -\theta\left(\alpha_1 F_1'' + \alpha_2 F_2''\right) > 0$. We can reduce eq. (43) to $\lambda^2 - F_1'\lambda - \Gamma = 0$. Eigenvalues are then $\lambda_1 = F_1' > 0$, $\lambda_2 = \frac{F_1' - \sqrt{(F_1')^2 + 4\Gamma}}{2} < 0$, and $\lambda_3 = \frac{F_1' + \sqrt{(F_1')^2 + 4\Gamma}}{2} > 0$.[25] The steady state is a saddle-point.

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[25]Note that strict concavity of production functions rule out nil eigenvalues.

References

Aghion, P.; Caroli, E., and García-Peñalosa, C. Inequality and Economic Growth: the Perspective of the New Growth Theories. *Journal of Economic Literature*, 37(4):1615–1660, 1999.

Asheim, G. B. Intergenerational Equity. *Annual Review of Economics*, 2(1):197–222, 2010.

Atkinson, A. B. and Harrison, A. J. *Distribution of personal wealth in Britain*. Cambridge Univ Press, 1978.

Atkinson, A. B. On the Measurement of Inequality. *Journal of Economic Theory*, 2(3):244–263, 1970.

d'Autume, A.; Hartwick, J. M., and Schubert, K. The zero discounting and maximin optimal paths in a simple model of global warming. *Mathematical Social Sciences*, 59(2):193–207, 2010.

d'Autume, A. and Schubert, K. Hartwick's rule and maximin paths when the exhaustible resource has an amenity value. *Journal of Environmental Economics and Management*, 56(3): 260–274, 2008a.

d'Autume, A. and Schubert, K. Zero discounting and optimal paths of depletion of an exhaustible resource with an amenity value. *Revue d'Économie Politique*, 118(6):827–845, 2008b.

Becker, R. A. On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households. *The Quarterly Journal of Economics*, 95(2):375–382, 1980.

Becker, R. A. and Foias, C. A Characterization of Ramsey Equilibrium. *Journal of Economic Theory*, 41(1):173–184, 1987.

Becker, R. A. and Tsyganov, E. N. Ramsey Equilibrium in a Two-Sector Model with Heterogeneous Households. *Journal of Economic Theory*, 105(1):188–225, 2002.

Bergson, A. A Reformulation of Certain Aspects of Welfare Economics. *The Quarterly Journal of Economics*, 52(2):310–334, 1938.

Cairns, R. D. and Long, N. V. Maximin: a direct approach to sustainability. *Environment and Development Economics*, pages 275–300, 6 2006.

Cairns, R. D. and Martinet, V. An environmental-economic measure of sustainable development. *European Economic Review*, 69:4–17, 2014.

Cairns, R. D.; Del Campo, S., and Martinet, V. Sustainability of an economy relying on two reproducible assets. *CESifo Working Paper Series*, (6314), 2016.

Caselli, F. and Ventura, J. A Representative Consumer Theory of Distribution. *American Economic Review*, 90(4):909–926, 2000.

Cass, D. Optimum Growth in an Aggregative Model of Capital Accumulation. *The Review of Economic Studies*, 32(3):233–240, 1965.

Chatterjee, S. Transitional dynamics and the distribution of wealth in a neoclassical growth model. *Journal of Public Economics*, 54(1):97–119, 1994.

Chiang, A. C. *Elements of dynamic optimization*. Mcgraw-Hill College, 1 1992.

Dasgupta, P. and Heal, G. The Optimal Depletion of Exhaustible Resources. *The Review of Economic Studies*, pages 3–28, 1974.

Diamond, P. A. The Evaluation of Infinite Utility Streams. *Econometrica: Journal of the Econometric Society*, 33(1):170–177, 1965.

Dixit, A.; Hammond, P., and Hoel, M. On Hartwick's Rule for Regular Maximin Paths of Capital Accumulation and Resource Depletion. *Review of Economic Studies*, 47(3):551–56, April 1980.

Fleurbaey, M. On sustainability and social welfare. *Journal of Environmental Economics and Management*, 71:34 – 53, 2015.

Fleurbaey, M. and Maniquet, F. *A Theory of Fairness and Social Welfare*. Cambridge University Press, first edition, 2011.

Garcìa-Peñalosa, C. Les inégalités dans les modèles macroéconomiques. *Revue de l'OFCE*, 153(4):105–131, 2017.

Garcìa-Peñalosa, C. and Turnovsky, S. J. Income Inequality, Mobility, and the Accumulation of Capital: The Role of Heterogeneous Labor Pproductivity. *AMSE working paper*, (16), 2012.

Hall, R. E. Intertemporal Substitution in Consumption. *Journal of Political Economy*, 96(2): 339–357, 1988.

Harrod, R. F. *Towards a Dynamic Economics*. Macmillan and Co., London, 1948.

Hartwick, J. M. Intergenerational equity and the investing of rents from exhaustible resources. *The american economic review*, pages 972–974, 1977.

Isaac, T. and Piacquadio, P. G. Equity and efficiency in an overlapping generation model. *Social Choice and Welfare*, 44(3):549–565, 2015.

Koopmans, T. C. Stationary Ordinal Utility and Impatience. *Econometrica: Journal of the Econometric Society*, 28(2):287–309, 1960.

Koopmans, T. C. On the Concept of Optimal Economic Growth. *Academiae Scientiarum Scripta Varia*, 28:225–300, 1965.

Kuznets, S. Economic Growth and Income Inequality. *The American Economic Review*, 45(1): 1–28, 1955.

Kuznets, S. and Jenks, E. Shares of Upper Income Groups in Savings. In *Shares of Upper Income Groups in Income and Savings*, pages 171–218. NBER, 1953.

Kverndokk, S.; Nævdal, E., and Nøstbakken, L. The trade-off between intra- and intergenerational equity in climate policy. *European Economic Review*, 69, 2014.

Lucas, R. E. On the Mechanics of Economic Development. *Journal of Monetary Economics*, 22(1):3–42, 1988.

Phelps, E. S. and Riley, J. G. Rawlsian Growth: Dynamic Programming of Capital and Wealth for Intergeneration "Maximin" Justice. *The Review of Economic Studies*, 45(1):103–120, 1978.

Pigou, A. C. *The Economics of Welfare*. Macmillan and Co., London, 1920.

Piketty, T. About Capital in the Twenty-First Century. *American Economic Review*, 105(5): 48–53, 2015.

Pitchford, J. D. Two State Variable Problems. *Essay 6 in Pitchford, J. D., and Turnovsky, S. J. (eds.) Applications of Control Theory to Economic Analysis, North-Holland, Amsterdam*, 101:127–154, 1977.

Ramsey, F. P. A Mathematical Theory of Saving. *The Economic Journal*, pages 543–559, 1928.

Samuelson, P. A. *Foundations of Economic Analysis*. Number 80 in Harvard Economic Studies. Cambridge - Harvard University Press, 1947.

Samuelson, P. A. and Solow, R. M. A Complete Capital Model Involving Heterogeneous Capital Goods. *The Quarterly Journal of Economics*, 70(4):537–562, 1956.

Sidgwick, H. *The Methods of Ethics*. Macmillan and Co., London, 7th edition, 1907.

Solow, R. M. A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, 70(1):65–94, 1956.

Solow, R. M. An almost practical step toward sustainability. *Resources policy*, 19(3):162–172, 1993.

Srinivasan, T. N. Optimal Savings in a Two-Sector Model of Growth. *Econometrica: Journal of the Econometric Society*, 32(3):358–373, 1964.

Stiglitz, J. E. Distribution of Income and Wealth Among Individuals. *Econometrica: Journal of the Econometric Society*, 37(3):382–397, 1969.

Swan, T. W. Economic Growth and Capital Accumulation. *Economic Record*, 32(2):334–361, 1956.

Uzawa, H. On a Two-Sector Model of Economic Growth. *The Review of Economic Studiese*, 29(1):40–47, 1961.

Uzawa, H. On a Two-Sector Model of Economic Growth II. *The Review of Economic Studies*, 30(2):105–118, 1963.

Uzawa, H. Optimal Growth in a Two-Sector Model of Capital Accumulation. *The Review of Economic Studies*, 31(1):1–24, 1964.

Withagen, C. and Asheim, G. B. Characterizing sustainability: The converse of Hartwick's rule. *Journal of Economic Dynamics and Control*, 23(1):159 – 165, 1998.